# Optimal trip planning in transport systems with random delays

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# The primary problem

Given:

1. a railway network and a timetable
2. a passenger and an origin and destination
3. the time the passenger would like to reach the destination
4. the probability that the passenger would like to reach the destination on time

Find:

1. the best route
2. the latest time the passenger should start from the origin

# More formally. . .

Let

- $\tau$ be the target arrival time
- $\epsilon$ be the probability that the passenger would like to reach the destination on time (failure probability)
- $t_d$ be the time the passenger departs from the origin
- $T_a$ be the time the passenger arrives at the destination (arrival time, a random variable)

Then the optimization problem is:

$$\begin{array}{rl} \text{max} & t_d \\ \text{subject to} & \text{Prob}[T_a > \tau] < 1 - \epsilon \end{array}$$

# Modelling train delay

In order to calculate Prob$[T_a > \tau]$, we will need to solve the following secondary problem:

- Given:
  1. A route
  2. A timetabled service for each train
  3. A model of the distribution of delays for each train
  4. A model of the distribution of starting time of the passenger
- Find:
  1. The probability distribution of the arrival time $T_a$ for the passenger at the destination

# Assumptions and notations

**Assumptions:**

- The departure times of any two trains are statistically independent
- The order of the departure for trains may vary due to delay
- When changing trains, passengers always catch the first train that departs to their next station on their chosen route

**Notation:**

$X_i \sim f_i$ probability distribution function (pdf)

$F_i$ cumulative distribution function (cdf)

$$F_i(t) = \int_{-\infty}^{t} f_i(x)dx$$

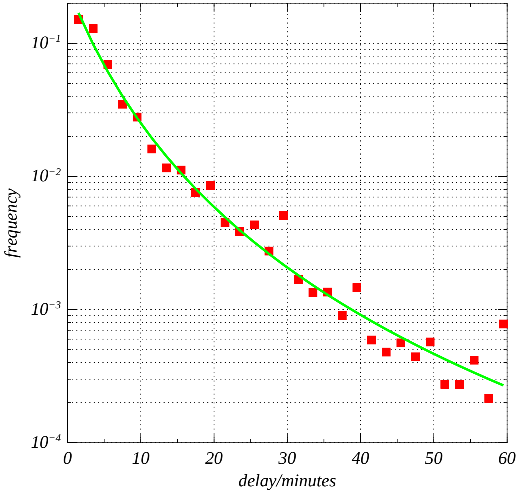
$$[\![x \geqslant t]\!] = \begin{cases} 1 & \text{if } x \geqslant t, \\ 0 & \text{otherwise} \end{cases}$$

# The $q$-exponential law

- Exponential law: $f_\beta(x) \propto \exp(-\beta x)$
- $e_{r,\beta}(x) := Z(1+\frac{\beta x}{r})^{-r}$, $\beta > 0$, $r > 1$
- $Z := \beta(1-\frac{1}{r})$
- mean $\mu := \frac{1}{\beta(1-\frac{2}{r})}$
- $\lim_{r\to\infty} e_{r,\beta}(x) = Z\exp(-\beta x)$
- small $r$ gives a power-law (long tail)
- The departure times for every trains can be modelled by $q$-exponential distribution by some parameters $\beta$ and $r$.

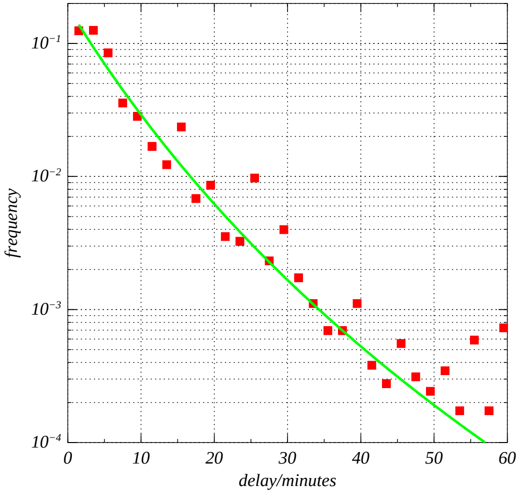
# Birmingham, all departures

$r = 3.8$
$\beta = 0.33$



# Coventry to Birmingham

$r = 9.0$
$\beta = 0.21$



# The discrete $q-$exponential model

- The discrete $q-$exponential model is more suited for the train model since the departure times of trains are considered in minutes
- Bins have length $\mathrm{d}t$, typically one minute
- The values for each bin are calculated using the CDF of the continuous model, i.e. $\int_a^{a+\mathrm{dt}} e_{r,\beta}(x)\ \mathrm{d}x$ where $a$ is the value of the bin
- The distribution is truncated; the departure delay cannot be $<0$ or greater than some maximum

# The subproblems

1. find candidate stochastically short paths
2. for each candidate short path, find probability distribution of arrival time
3. optimize over short paths

# Stochastically short paths on a graph

- Given a graph with RVs as edge weights and two nodes, we could:
  - minimize expected time to travel between the nodes
  - find a route which maximizes the probability that it is shortest
  - find the route of shortest mean time, subject to some condition on the variance
  - ...

# The problem formalized

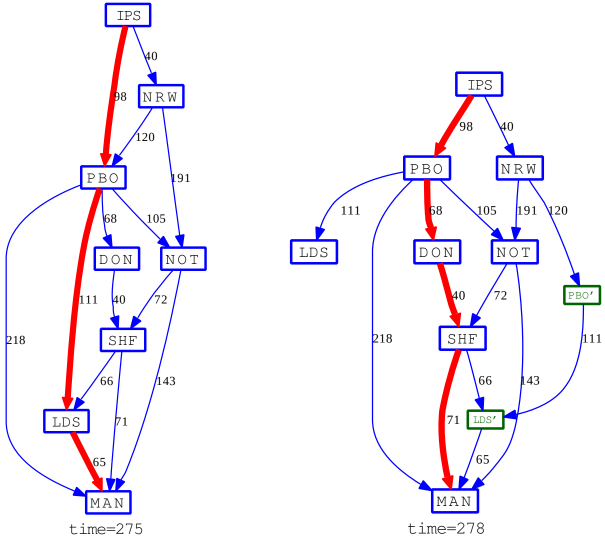
Given:

- a weighted digraph $g$
- a timetable $\mathsf{TT}(n_0, n_1)$ for each arc $(n_0, n_1) \in g$
- a final arrival time $T_a$, and parameters $\tau > 0, \epsilon > 0$

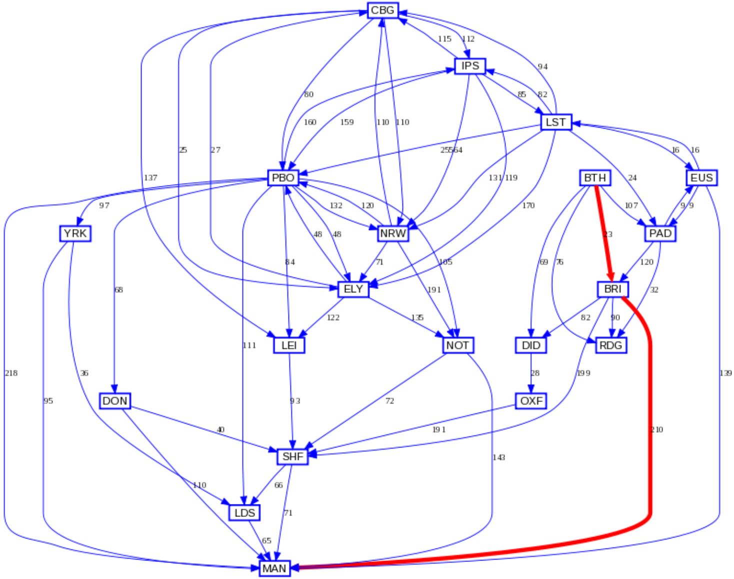
To find:

- a route $\rho$ and maximal departure time $t$ such that Prob[arrival after $\alpha + \tau$] $< \epsilon$

# Short paths in a weighted digraph

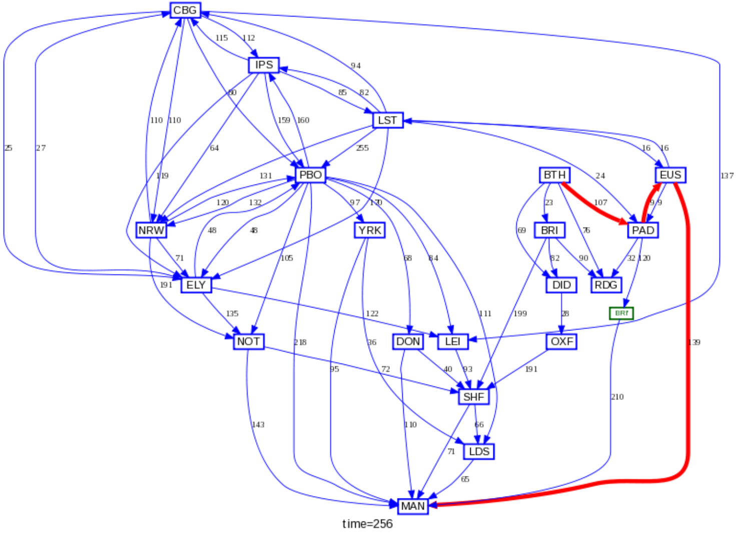


# Bath to Manchester, shortest mean time




time=233

# Bath to Manchester, second shortest mean time



# Transformation using a kernel

- The departure time of the passenger at the initial station is modelled as a probability distribution
- We need to compute the probability distribution model for the arrival time of the passenger at the next station

$$A_0 \xrightarrow[K_0]{} A_1 \xrightarrow[K_1]{} \dots \xrightarrow[K_{r-1}]{} A_r$$

Here:

- $A_i$ is the probability distribution of arrival time at station $i$
- $K_i$ is the mapping between input and output distributions; we will call it the *kernel*

# Aside: order statistics

Let $X_i \sim f$, $i = 0, \ldots, n-1$ be iid. Define $X_{(0)}$ to be the minimum $X_i$. We compute the pdf $f_{(0)}$ of $X_{(0)}$:

$$\begin{aligned}
f_{(0)}(x)\mathrm{d}x &= \mathrm{Prob}[X_{(0)} \in \mathrm{d}x] \\
&= \mathrm{Prob}[\text{one } X_i \in \mathrm{d}x, \text{ others } > x] \\
&= \sum_i \mathrm{Prob}[X_i \in \mathrm{d}x, \underset{j \neq i}{X_j} > x] \\
&= \sum_i \mathrm{Prob}[X_i \in \mathrm{d}x] \mathrm{Prob}[\underset{j \neq i}{X_j} > x] \\
f_{(0)}(x) &= \sum_i f(x) \prod_{j \neq i} (1 - F(x)) \\
&= n f(x) (1 - F(x))^{n-1}
\end{aligned}$$

# The continuous time model

Let $K_{(0|t)}(x)$ be the probability density of departing at time $x$ given that the passenger arrives at time $t$.

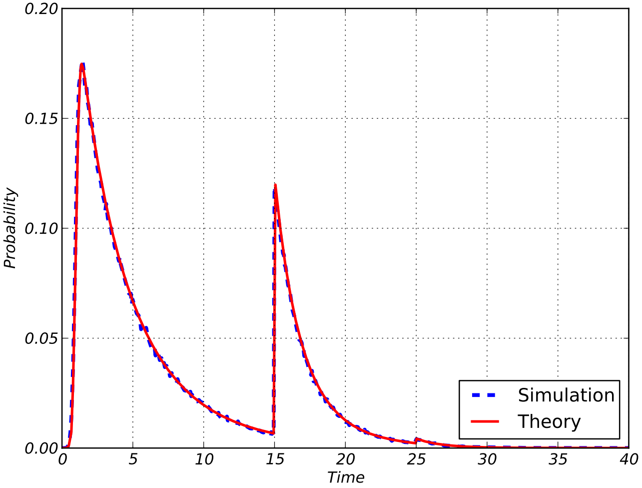
$$\begin{aligned}
K_{(0|t)}(x)\mathsf{d}x &= [\![x>t]\!]\,\mathsf{Prob}[X_{(0|t)}\in\mathsf{d}x] \\
&= [\![x>t]\!]\,\mathsf{Prob}[\text{one } X_i\in\mathsf{d}x,\ \text{others } >x \text{ or } \leqslant t] \\
&= [\![x>t]\!]\sum_i \mathsf{Prob}[X_i\in\mathsf{d}x,\ \underset{j\neq i}{X_j}>x \text{ or } \leqslant t] \\
&= [\![x>t]\!]\sum_i \mathsf{Prob}[X_i\in\mathsf{d}x]\,\mathsf{Prob}[\underset{j\neq i}{X_j}>x \text{ or } \leqslant t] \\
K_{(0|t)}(x) &= [\![x>t]\!]\sum_i f_i(x)\prod_{j\neq i}(1-F_j(x)+F_j(t))
\end{aligned}$$

# Integral transforms

$$A_n(x) = \int_{-\infty}^{\infty} A_{n-1}(t) K_{(n-1|t)}(x) \mathrm{d}t, \; n = 1, 2, \ldots$$

- Exact results (and agree with simulation)
- Integrals cannot be done analytically
- The amount of computer time needed to calculate $A_i(x)$ increases dramatically as $i$ increases

# Continuous model simulation for catching one train



# The discrete time model

Let $T$ be the set of all trains, $t$ be the time the passenger arrives at the station and $x$ be the time the passenger departs.

$$\begin{aligned} K_{(0|t)}(x) &= [\![x>t]\!]\,\mathrm{Prob}[X_{(0|t)}=x] \\ &= [\![x>t]\!]\,\mathrm{Prob}[\text{at least one } X_i=x,\ \text{others } >x \text{ or } \leqslant t] \end{aligned}$$

Let $T$ be the set of trains departing after time $t$; then

$$\begin{aligned} K_{(0|t)}(x) &= [\![x>t]\!] \sum_{\emptyset\neq S\subseteq T} \mathrm{Prob}[\underset{i\in S}{X_i}=x,\ \underset{j\notin S}{X_j}>x \text{ or } \leqslant t] \\ &= [\![x>t]\!] \sum_{\emptyset\neq S\subseteq T} \mathrm{Prob}[\underset{i\in S}{X_i}=x]\,\mathrm{Prob}[\underset{j\notin S}{X_j}>x \text{ or } \leqslant t] \\ &= [\![x>t]\!] \sum_{\emptyset\neq S\subseteq T} \prod_{i\in S} f_i(x) \prod_{j\notin S} (1-F_j(x)+F_j(t)) \end{aligned}$$

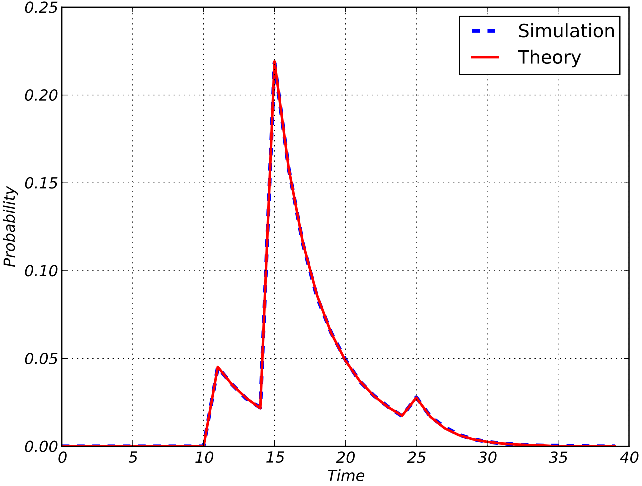
# Matrix multiplication of kernels

Let $K_0 = (K_{(0|j)}(i))$, then

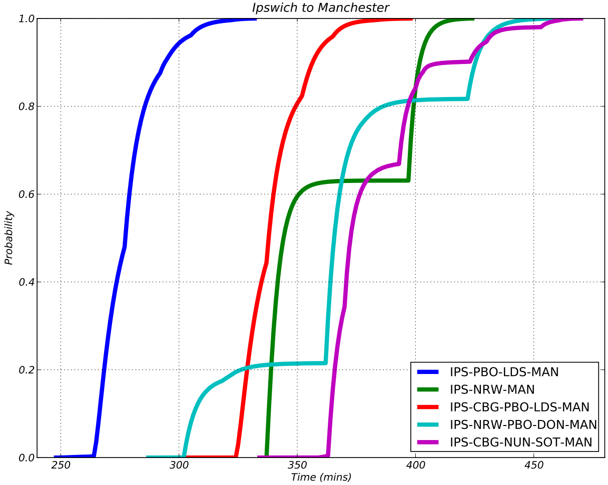
$$A_1 = K_0 A_0$$
$$A_2 = K_1 A_1 = K_1 K_0 A_0$$
$$\ldots$$
$$A_n = K_{n-1} \ldots K_0 A_0$$

- Kernel and the distribution of arrival time can be considered as matrix and vector
- Finding the distribution of the arrival time at the next station is exactly the same as matrix multiplication

# Discrete model simulation for catching one train



# CDF for journey times from Ipswich to Manchester



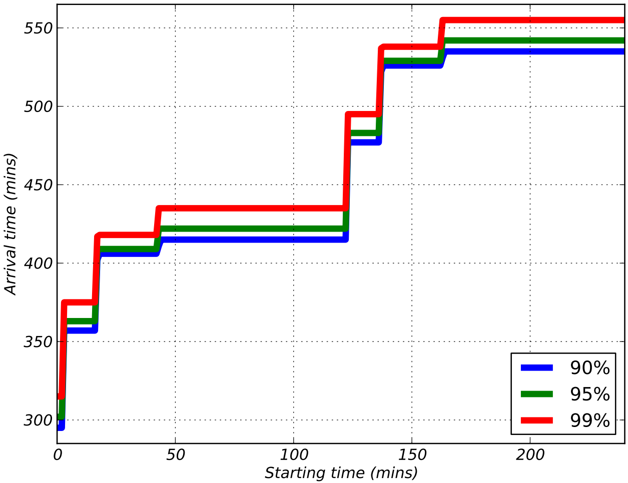
# Finding the optimized departure time

- From the CDF we can calculate the time a passenger will arrive the destination at a given probability for a fixed departure time
- We would like to find the latest departure time such that the passenger will arrive the destination on time at a probability given by the passenger
- The optimized departure can be found by e.g. a bisection method

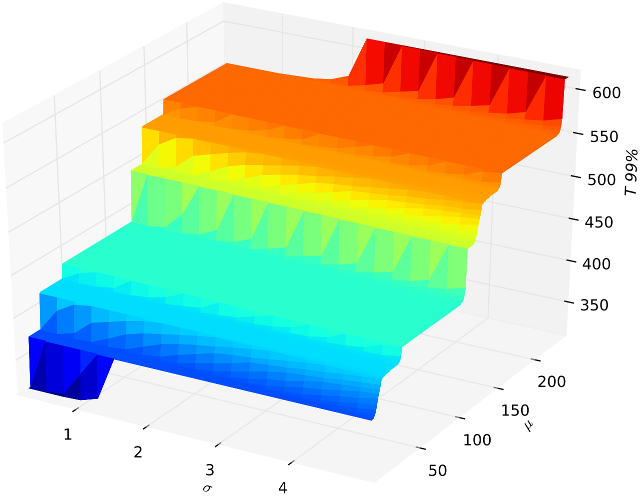
# Bisection method

1. Obtain two departure times, $x$, $y$ such that $x < y$ and obtain the arrival time $t_x$, $t_y$ using the CDF of departure time $x$ and $y$ respectively.
2. If the arrival time set by the passenger $T$ is not between $t_x$ and $t_y$, then obtain a new $x$ and $y$ accordingly
3. Find the midpoint $z$ of $x$ and $y$
4. Calculate the CDF for departure time $z$ and obtain the arrival time $t_z$
5. If $t_x \leqslant T \leqslant t_z$, set $y := z$, otherwise set $x := z$
6. Repeat the process until $y - x$ is sufficiently small

# Arrival time for journey from Ipswich to Manchester with different starting time



# Arrival time for journey from Ipswich to Manchester with starting times as Gauss function



# References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# Gray code method

- The Gray code method is a method of generating a list of subsets such that adjacent subsets in the list differ only by a single element
- Using the Gray code method, we can generate all the subsets with the same cardinality
- Since after each iteration the new set generated differs by a single element from the previous set, we can use the product obtained from the previous iteration to calculate the next product
- There is no need to multiply all the products again; we only need to divide (element taken out from the previous set) and multiply (element put in the new set) once and we will obtain the next product

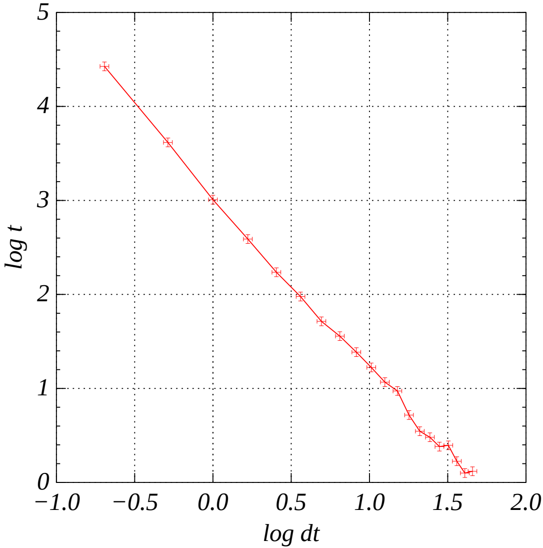
# Changing the length of each interval

- By default, the probabilities are calculated for every minute, i.e. each interval is one minute.
- For a long journey, the amount of calculations will increase significantly.
- The passengers may prefer to know the probability of arriving at their destinations at some arbitrary interval.

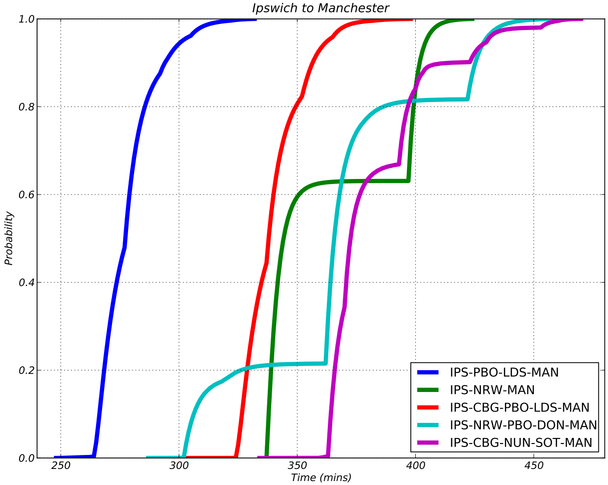
Let $t$ be the CPU time for the calculation of each journey and $dt$ be the length of each interval, then

- $\frac{\log t}{\log dt} \sim 2$ or $t \sim dt^2$

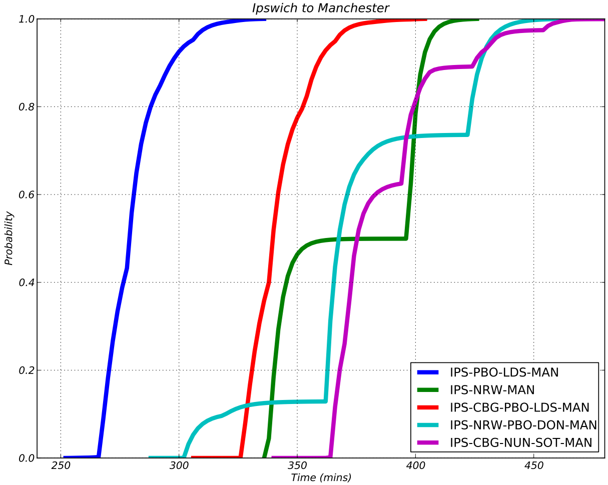
# CPU time needed with different interval length



# CDF for journey from Ipswich to Manchester with interval length $1$ when departure at time $0$



# CDF for journey from Ipswich to Manchester with interval length $2$ when departure at time $0$



# Calculating $K_{(0|t)}(x)$

- The total numbers of non-empty subsets of $T$ is $2^m - 1$ where $m$ is the cardinality of $T$
- The number of operations is exponential
- Some or maybe most of the terms in the sum are zero
- The number of calculations can be reduced if we can find these zero terms
- The number of operations can also be reduced by reducing some degree of accuracy to the probability

# Finding the zero terms

Let $S$ be any non-empty subset of $T$ and

$$f^S(x) = \prod_{i \in S} f_i(x) \text{ and } g^S(x,t) = \prod_{j \notin S} g_j(x,t) = \prod_{j \notin S} (1 - F_j(x) + F_j(t)).$$

Then

$$K_{(0|t)}(x) = \sum_{\emptyset \neq S \subseteq T} f^S(x) g^S(x,t).$$

- The product is zero if any factor in either $f^S(x)$ or $g^S(x,t)$ is zero
- Since the order of multiplication and addition will not affect the result, we can sort $f_i(x)$ in descending order and relabel the indices

- Find all the $i$ such that $f_i(x) \neq 0$ and let the collection of indices be the set $U$
- Find all the $j \in U$ such that $g_j(x,t) \neq 0$ and let the collection of indices be the set $V$

Then the set $T$ can be partitioned into the following subsets, $V$, $U-V$ and $T-U$ and we can obtain the following table:

| $i \in$ | $V$ | $U-V$ | $T-U$ |
|---|---|---|---|
| $f_i(x)$ | $>0$ | $>0$ | $=0$ |
| $g_i(x,t)$ | $>0$ | $=0$ | $\geqslant 0$ |

From the table, we will have the following observations:

- If $\exists i \in T - U$ such that $g_i(x,t) = 0$, then either $f^S(x)$ or $g^S(x,t)$ is zero for all $S$ and hence $K_{(0|t)}(x) = 0$
- If $f^S(x)g^S(x,t) \neq 0$, it immediately implies that $U - V \subseteq S$
- Similarly, if $f^S(x)g^S(x,t) \neq 0$, it immediately implies that $T - U \subseteq T - S$
- Although the number of operation is still exponential, the total number of subsets is reduced

# Controlling the error

- Without loss of generality, we can assume that $f^S(x)g^S(x,t) \neq 0, \ \forall S$
- First consider all possible subsets of $T$ with cardinality one and calculate the sum, i.e. $\sum_{|S|=1} f^S(x)g^S(x,t)$
- Calculate the upper bounds by consider the following:

$$\text{upper bound} = \begin{array}{l} \text{remaining number of subsets } \times \\ \text{upper bound on products } f^S(x)g^S(x,t), \, |S|>1 \end{array}$$

- If the upper bound is less than some satisfactory value, i.e. $\epsilon$ (absolute error) or $\epsilon \times$sum (relative error), then the sum is accepted
- Otherwise, consider all subsets with cardinality one and two and repeat the process until the upper bound is less than $\epsilon$

# Increasing the speed of matrix multiplication

- The number of rows $(i)$ needed is not known in general
- While calculating $d_{l+1}$, instead of calculating the whole matrix $K_l$ by using random number of rows, we will calculate the first row of $K_l$ and find the first element of $d_{l+1}$
- Then we will calculate the second row of $K_l$, find the second element of $d_{l+1}$ and repeat the process.
- Since $d_{l+1}$ is a probability distribution, therefore the sum of all element in $d_{l+1}$ is equal to $1$
- Using this fact, we will calculate the sum after every step and we will stop when the sum is equal or relatively close to $1$
- This method not only solve the problem of not knowing the number of rows required, it also uses less memory since no large matrix is calculated and saves compution time, as only required rows are calculated.