FEIGENBAUM SCALING
IN
DISCRETE DYNAMICAL SYSTEMS

by
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Abstract

In this thesis I study some generalizations of Feigenbaum's discovery of scaling
in families of nonlinear discrete dynamical systems. I first make a precise com-
putation of the Feigenbaum constants for unimodal real maps. This allows me
to study number-theoretic properties of these quantities. I then make a detailed
study of the asymptotic limits of the feigenvalues as the degree of the maximum
of the unimodal map goes to infinity. The result is the first precise computation
of the asymptotic feigenvalues. I also generalize the Feigenbaum scaling law to
compute corrections to scaling in real maps. In the next chapter I consider scaling
in complex analytic maps. The results here include a complete classification of
the possible feigenvalues up to degree eight. I construct a complex version of the
thermodynamic formalism, which allows a computation of the Hausdorff dimension
of attractors of universal functions. I also study scaling in the area of hyperbolic
components of the Mandelbrot set. The next chapter concerns circle maps, includ-
ing scaling on the boundaries of Siegel disks of complex analytic maps. I compute
corrections to scaling in circle maps, and the asymptotic limit of the Feigenbaum-
Kadanoff-Shenker scaling constant, followed by a discussion of Manton-Nauenberg
scaling and corrections to scaling. I next move to higher dimensions with a study
of scaling in two-torus maps. This requires some introductory discussion of cubic
number field theory. This chapter contains the first evidence for scaling in two-
torus maps. I give next a detailed discussion of iteration of quaternion maps. This
is an attempt to see which properties of complex analytic maps continue in the
four-dimensional space of quaternions. In particular, I classify all regularly iter-
able linear quaternion maps. A final chapter describes the design of algorithms for
the solution of functional equations which were used throughout the thesis.
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Preface

Versions of some parts of this work were published during the writing of this thesis. These are § 2.1, § 2.3, § 3.1, and parts of Chapters 6 and 7.

In my work on solutions of Feigenbaum's functional equation with an essential singularity, I collaborated with George Szekeres of the University of New South Wales and Tony Dixon of the University of Western Australia. The formulation of the domains for the Abel functional equation was done by George Szekeres, I designed all the computer programs used to solve the functional equations, and was assisted in the implementation and running of these programs by Tony Dixon. A version of this section has been submitted for publication as a joint paper by the three of us.

In my work on feigenvalues of mandelsets I collaborated with my supervisor, Colin Thompson, and Reinout Quispel of La Trobe University. A version of § 3.1 was published as a joint paper by the three of us.

In my work on quaternion maps I collaborated with Stephen Bedding of La Trobe University. In § 6.5.1 the initial idea of using Schur forms was mine, but Stephen Bedding did most of the Maple calculations exploiting this idea.

I wrote this thesis on Pentium 90 hardware running linux 1.2.13 with $\text{\LaTeX} 2_\varepsilon$. Since this is all free software, I must thank the authors: Linus Torvalds for linux, and Donald Knuth and the $\text{\LaTeX}$ team for $\text{\LaTeX} 2_\varepsilon$. I designed the main text font myself, based on Knuth's concrete font. The maths font is euler roman, except for the digits $0, 1, \ldots, 9$. The bibliographic reference for this thesis is [Briggs, 1997]. In order to satisfy University of Melbourne regulations, I declare that according to `detex *tex | wc`, this thesis contains about $5 \times 10^4 < 10^5$ words.

This version of January 11, 2001 includes minor corrections made since the final submitted version. These are marked with marginal changebars.

Acknowledgements

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Chapter 1

Introduction

Macbeth: ' ... *we delight in physics* ... '

Macbeth, act 2, scene 3

1.1   History

Feigenbaum scaling was discovered independently and almost simultaneously by Grossmann and Thomae [Grossmann and Thomae, 1977] and Feigenbaum [Feigenbaum, 1978]. Curiously, despite being a purely mathematical phenomenon, it was discovered by theoretical physicists, and though mathematicians contributed to the subsequent development of the theory, most of the resulting papers have appeared in the physics literature.

In the late 1970s the discovery of Feigenbaum scaling generated much excitement, which lasted for about ten years. This was part of the parallel development of nonlinear dynamical systems theory. By the late 1980s most physicists appeared to have decided that the subject was completely understood, and little more remained to be done. I hope to show otherwise in this thesis. The term 'universality' is often used in this context (from the analogy with universality of phase transitions in statistical mechanics), but I avoid this over-enthusiastic claim. Rather, we can claim that Feigenbaum scaling is *generic*, meaning that it occurs in all systems except those in which it does not occur. The subject has been well surveyed several times; for example, [Vul, Sinai and Khanin, 1984; Cvitanović, 1989; Argyris, Faust and Haase, 1995]).

In this thesis, I aim to see how far the concept of Feigenbaum scaling can

be generalized. Some of the highlights are the most precise determination of the real Feigenbaum scaling constants (§ 2.1) and their asymptotic limits (§ 2.4); the classification of complex solutions to the Feigenbaum functional equation (§ 3.2); the clarification of the nature of Manton-Nauenberg scaling (§ 4.4); the first detailed discussion of quaternion iteration theory (Chapter 6), and the first evidence for scaling in two-torus maps (§ 5.2).

1.2   What is Feigenbaum scaling?

Throughout this thesis, I consider families of discrete dynamical systems defined by iteration of maps f_μ on a space $\mathbb{X}$:

$$f_\mu : x \longmapsto f_\mu(x). \tag{1.1}$$

Here $\mathbb{X}$ will stand for the reals $\mathbb{R}$, the plane $\mathbb{R}^2$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, the circle $\mathbb{T}^1$ or the two-torus $\mathbb{T}^2$. The maps f_μ will be assumed sufficiently differentiable or even analytic, as required. The real, complex, or quaternion number μ is considered the control parameter. In experiments modelled by equation (1.1), μ is varied slowly compared to the rate of iteration of f_μ. See [Briggs, 1987] for examples of such experiments. In applications this family of maps f_μ may arise from a Poincaré section or stroboscopic map of a flow. The sequence $\langle x_0, x_1, x_2, \ldots \rangle$, where $x_n = f_\mu(x_{n-1}), n = 1, 2, 3, \ldots$, is called the *orbit* of the *seed* x_0. I also denote the nth iterate x_n by $f^{<n>}(x_0)$.

An orbit with $x_n = x_0$ for some integer $n > 0$ (the *period*) is called an n-cycle. In the case of $\mathbb{X} = \mathbb{R}$ or $\mathbb{C}$, every n-cycle has a *stability* ρ defined by

$$\rho = \prod_{i=0}^{n-1} f_\mu'(x_0). \tag{1.2}$$

We call orbits with stability satisfying $|\rho| < 1$ *stable*, and *superstable* if $\rho = 0$. In the case of maps on a space of dimension greater than one, we replace f' in equation (1.2) by the Jacobian matrix, and require for stability that all eigenvalues of the stability matrix ρ have modulus less than unity. Most of Feigenbaum scaling theory is concerned with the behaviour of superstable orbits; in particular, with the rate of variation of μ as the period n goes to infinity. For this to make sense, we need to study a family of maps which possesses a complete sequence of period-doubling bifurcations, so that we may find an infinite sequence of parameters values

$\{\mu_0, \mu_1, \mu_2, \ldots\}$ such that a stable cycle of period 2^k of f_μ bifurcates to a stable 2^{k+1}-cycle at $\mu = \mu_k$. We will also consider n-furcations for $n = 3, 4, 5, \ldots$ later. The period-doubling process by which this takes place is described in all elementary texts on dynamical systems (for example [Arrowsmith and Place, 1990]). The standard example on the real line is the family $f_\mu(x) = \mu x(1 - x)$. For this family of maps, Feigenbaum made the original discovery [Feigenbaum, 1978] that the limit

$$\delta \equiv \lim_{j \to \infty} \frac{\mu_j - \mu_{j-1}}{\mu_{j+1} - \mu_j} \tag{1.3}$$

exists. Feigenbaum also computed an approximate value 4.6692 for this limit. Another way of saying this is that

$$\Delta \mu_k \equiv \mu_k - \mu_{k-1} \sim C\delta^{-k}$$

for some constants C and δ. One may replace the bifurcation parameter values μ_k by superstable parameter values without change to the limit. In fact, this is much preferable for numerical work. Feigenbaum showed additionally that there exists an orbit scaling: If ϕ_k is the value of the nearest cycle element to zero in the 2^k-cycle, then

$$\alpha \equiv \lim_{k \to \infty} \frac{\phi_k}{\phi_{k+1}} \tag{1.4}$$

exists, and is about -2.503.

These constants α and δ have become known familiarly as 'feigenvalues'. This scaling can be seen in Figure 1.1, which shows the bifurcation diagram (that is, the stable cycles as a function of the map parameter) for the family $x \mapsto 1 - ax^2$, with a logarithmic a scale. Feigenbaum found that all quadratic maps of the real line possessed the same constant $\delta = 4.6692$, and he also found $\delta \approx 7.28$ for quartic maps.

Originally, physicists had hoped that these so-called universal scaling constants would prove useful in experimental situations, in that one may predict the next bifurcation value of μ given any three successive values. Unfortunately, this hope has not been fulfilled in practice, due to the difficulty of seeing more than three or four bifurcations in typical experiments. These experiments have now been performed in many fields: fluid dynamics [Buzug, von Stamm and Pfister, 1993]; electronics [Briggs, 1987]; chemistry [Epstein, 1983], and mechanical engineering [Slivsgaard and True, 1994]. It is unavoidably the case that the relatively large value of δ means that the bifurcations are lost in the experimental noise very quickly. Nevertheless, the whole subject of Feigenbaum scaling has remained of great interest from a mathematical point of view.
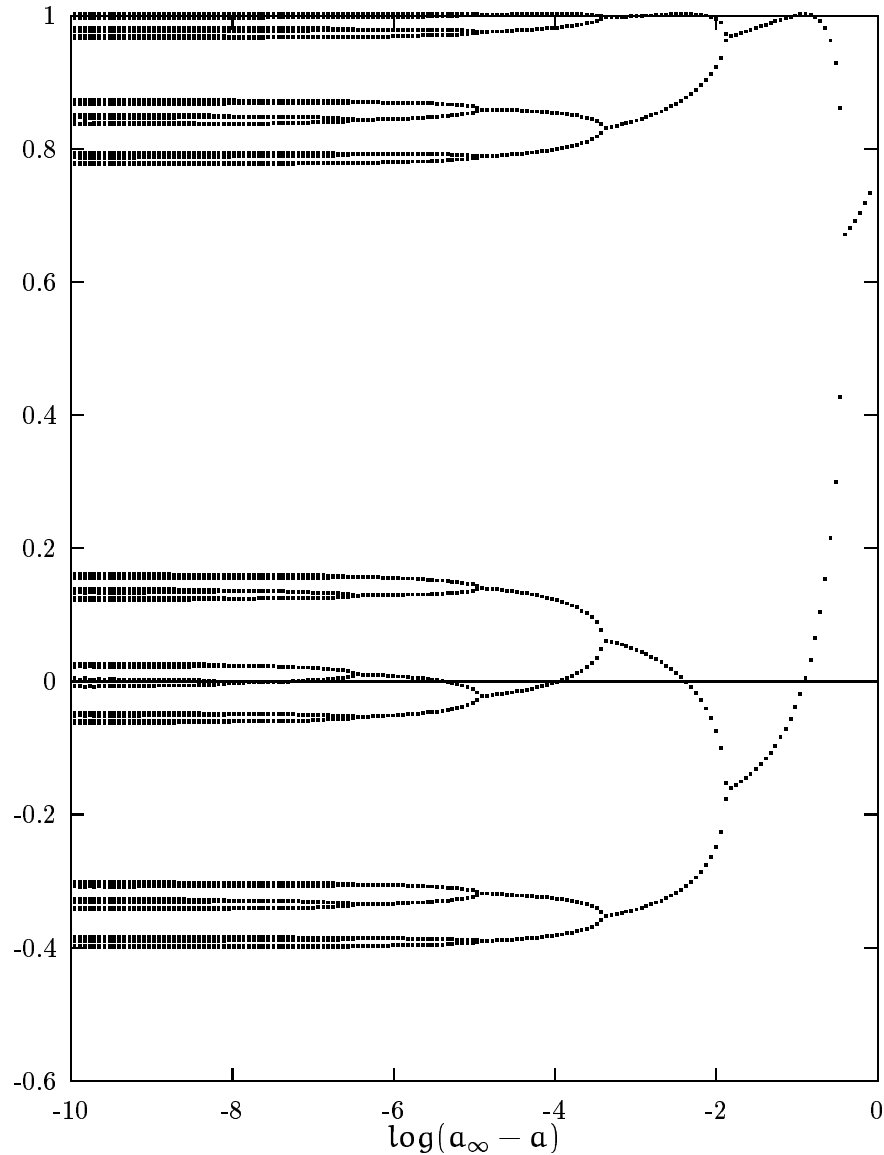
Figure 1.1: The figtree logged

1.3 The renormalization theory

Feigenbaum's achievement in his early papers was not simply to discover the scaling behaviour, but to also give an explanation in terms of the period-doubling operator. This is the operator T defined by its action on a function f:

$$f(x) \rightarrowtail (Tf)(x) \equiv \alpha f \circ f(\alpha^{-1}x), \qquad (1.5)$$

where $\alpha^{-1} \equiv f(0)$. He then proposed that the scaling behaviour could be understood in terms of a function fixed by T, so that

$$\alpha g \circ g(\alpha^{-1}x) = g(x). \qquad (1.6)$$

This has become known as Feigenbaum's functional equation (FFE), and T is the period-doubling *renormalization operator*. It is also often known as the Cvitanović-Feigenbaum equation, in recognition of the significant contribution of Predrag Cvitanović in the early stages of this work. The idea was that for large integers k, appropriately scaled versions of the functions $f_{\mu_k}^{<2^{k-1}>}$ and $f_{\mu_{k+1}}^{<2^k>}$ should approach each other, at least near zero, for any suitable family f_μ; and that the limiting function

$$g(x) = \lim_{k \to \infty} g_k(x)$$

where

$$g_k(x) = (T^k f)(x) = \alpha^k f_{\mu_k}^{<2^k>}(\alpha^{-k} x),$$

should be independent of the family f_μ and should satisfy the functional equation (1.6). This explanation turned out to be correct, and various rigorous results about the solutions of equation (1.6) have since been proven [Campanino, Epstein and Ruelle, 1982; Lanford, 1980; Epstein and Lascoux, 1981; Lanford, 1982; Vul et al., 1984; Koch, Schenkel and Wittwer, 1994]. Since all scaling properties of period-doubling in a given dynamical system can be deduced from the appropriate solution g of the FFE, each solution g is appropriately called a *universal function*. Also important in this description is the linearization DT_g of T about its fixed point g, whose action on a function h is given by

$$(DT_g h)(x) = \alpha g' \circ g(\alpha^{-1} x) h(\alpha^{-1} x) + \alpha h \circ g(\alpha^{-1} x) +$$
$$[(\alpha^2 - 1)h(0) + \alpha h(1)][x g'(x) - g(x)],$$

the last term here arising from the dependence of α on f. Feigenbaum proposed an explanation of δ as the largest eigenvalue of DT_g greater than one, which turns out to be unique for the case of real quadratic maps. Other technical details of this theory have been fully described by Vul et al. [Vul et al., 1984]. In this thesis I will largely be concerned with various generalizations of equation (1.6).

1.4 Other developments

Here are some other recent developments in Feigenbaum scaling theory, less directly related to this thesis.

- In [Kawai and Tye, 1984], Kawai and Tye discuss scaling in maps which are not analytic at their maximum; also, maps with asymmetric maxima.

- In [Vul et al., 1984], Vul, Sinai and Khanin study the relationship Feigenbaum scaling and the thermodynamic formalism of dynamical systems theory due to Ruelle and others. This paper also gives one the best surveys of the renormalization theory approach.

- In [Hu and Mao, 1987], Hu and Mao study the transition to chaos in higher dimensions; in particular, in $\mathbb{R}^2$ and $\mathbb{R}^4$.

- In [Post, 1991], Post studied the scaling of periodic windows of unimodal maps.

- In [Warner and Delbourgo, 1991], Warner and Delbourgo discuss the morphology of the period-doubling universal function, that is, the form of its extrema and its asymptotic behaviour.

- In [Roberts and Quispel, 1992], Roberts and Quispel studied reversible maps, especially of the plane, and discuss scaling in such maps in their section 5.2. Reversible maps are those which can be written as the composition of two involutions; that is, two maps which are each compositional square roots of the identity map.

- In [Zaks, 1994], Zaks studies what he calls double exponential scaling in sequences of m-tuplings with non-integer m for unimodal mappings. He makes an interesting analogy with the circle map golden-mean scaling (see § 4 of this thesis) to make a generalization of the definition of m-tupling to non-integer m.

- In [Wells and Overill, 1994], Wells and Overill discuss the extension of the real solution of the Feigenbaum functional equation to the complex plane.

- In several papers, [Kim and Hu, 1988; Kim, 1994; Kim, 199?], Sang-Yoon Kim has studied scaling in higher dimensional maps. In [Kim and Hu, 1988], he looks at period n-tupling in area-preserving maps. In [Kim, 1994; Kim, 199?], he studies four-dimensional maps.

The subject of Feigenbaum scaling in maps of the plane $\mathbb{R}^2$ is vast and has not been satisfactorily surveyed in the literature. As it is not directly relevant to this thesis, I will here just summarize the most significant work:

- Derrida and Pomeau [Derrida and Pomeau, 1980] used a low-order renormalization method with the Hénon map

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} 1 + y - ax^2 \\ bx \end{bmatrix}$$

  to obtain approximate values of δ.

- In [Benettin, Cercignani, Galgani and Giorgilli, 1980] and [Benettin et al., 1980] the authors obtained numerical confirmation of the Derrida and Pomeau result ($\delta \approx 8.72$) via direct studies of plane maps. They also obtained estimates of the two orbit scaling parameters $\alpha \approx 4.02$ and $\beta \approx$ -16. These values were also obtained by Bountis [Bountis, 1981].

- [Collet, Eckmann and Koch, 1981a] formulated a two-dimensional analog of the Feigenbaum functional equation with a diagonal matrix taking the place of α. They solved this with two-variable power series, confirming again the numerical values of δ, α and β.

- [Greene, MacKay, Vivaldi and Feigenbaum, 1981] studied the de Vogalaere map

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} y_k + Cx_k + x_k^2 \\ x_k - Cx_{k+1} - x_{k+1}^2 \end{bmatrix}$$

  and applied a renormalization scheme. They obtained estimates of the scaling constants accurate to about eight decimal places. These were in agreement with the Hénon map values.

- In two papers [Eckmann, Koch and Wittwer, 1982; Eckmann, Koch and Wittwer, 1984], Eckmann, Koch and Wittwer give a rigorous proof of the existence of a solution to the two-dimensional Feigenbaum functional equation

$$\Lambda^{-1} \circ \Phi \circ \Phi \circ \Lambda = \Phi,$$

  where Λ is a two-by-two scaling matrix. More recently, Davie [Davie, 1995] has given another proof of this result, this time without computer assistance.

- Widom and Kadanoff [Widom and Kadanoff, 1982] have used a generating function to study equation (1.4).

- Finally, Hauser, Curado and Tsallis [Hauser, Curado and Tsallis, 1985] have discussed generalized Hénon maps with the power 2 replaced by a larger integer z. They find a dependence of δ on z.

- [Reick, 1992] discussed correction to scaling in two-dimensional maps.

Chapter 2

Feigenbaum scaling in real maps

Falstaff: *'O, thou hast damnable iteration ... '*

King Henry IV part 1, act 1, scene 2

2.1 A precise calculation of the Feigenbaum constants

The Feigenbaum constants were first discovered in the theory of iteration of real functions, and I will begin by calculating to high precision the constants α and δ associated with period-doubling bifurcations for maps with a single maximum of order d, for d between two and twelve. Multiple-precision floating-point techniques are used to find a solution of the Feigenbaum functional equation, and hence the constants.

2.1.1 Background

I will consider the iteration of the family of maps

$$f_{\mu,d}(x) = 1 - \mu|x|^d, \qquad \mathbb{R} \ni d \geqslant 1, \qquad (2.1)$$

that is, sequences generated from the seed x_0 by

$$x_{i+1} = f_{\mu,d}(x_i), \qquad i = 0, 1, 2, \ldots . \qquad (2.2)$$

As described in Chapter 1, Feigenbaum [Feigenbaum, 1979] observed that there exist bifurcations in the set of limit points of the sequence (2.2) (that is, in the set

Reference	d	$-\alpha$
[Feigenbaum, 1979; Feigenbaum, 1980a]	2	2.502907876
[Feigenbaum, 1979; Feigenbaum, 1980a]	2	2.50290787509589284
[Christiansen et al., 1990]	2	2.502907875095892822283390287
	2	2.5029078
	4	1.690
	6	1.467
	8	1.358
	10	1.292
[Delbourgo and Kenny, 1986]	2	
	3	1.93
	4	1.69
	5	1.56
	7	1.41
	10	1.29

Table 2.1: Imprecise values of Feigenbaum's α in the earlier literature

of all points which are the limit of some infinite subsequence) as the parameter μ is increased for fixed d. Roughly speaking, if the sequence (2.2) is asymptotically periodic with period p for a particular parameter value μ (that is, there exists a stable p-cycle), then as μ is increased, the period will be observed to double, so that a stable 2p-cycle appears. I denote the critical value of μ at which the 2^j cycle first appears by μ_j.

Feigenbaum also conjectured that there exist certain 'universal' scaling constants associated with these bifurcations. Specifically,

$$\delta(d) \equiv \lim_{j \to \infty} \frac{\mu_j - \mu_{j-1}}{\mu_{j+1} - \mu_j}$$

exists, and δ about 4.669. Similarly, if ϕ_j is the value of the nearest cycle element to zero in the 2^j cycle, then

$$\alpha(d) \equiv \lim_{j \to \infty} \frac{\phi_j}{\phi_{j+1}}$$

exists, and α is about -2.503. The conjecture for the case d = 2 was proven by Lanford in 1982 [Lanford, 1982], and for $d \leqslant 14$ by Epstein in 1986 [Epstein, 1986]. Some numerical results in the literature are given in Tables 2.1 and 2.2. The most precise known values (for d = 2 only) before my calculation were those of [Christiansen, Cvitanović and Rugh, 1990].

I have described some examples of physical systems in which α and δ are relevant in [Briggs, 1987]. Despite the theoretical and applied interest of these numbers, little is known about their mathematical properties; for example, whether

Reference	d	δ
[Feigenbaum, 1979; Feigenbaum, 1980a]	2	4.6692
[Feigenbaum, 1979; Feigenbaum, 1980a]	2	4.6692016091029909
[Christiansen et al., 1990]	2	4.6692016091029906718532038
	2	4.6692016
	4	7.29
	6	9.30
	8	10.948
	10	12.37
[Delbourgo and Kenny, 1986]	2	4.67
	3	6.08
	4	7.29
	5	8.35
	7	10.2
	10	12.3

Table 2.2: Imprecise values of Feigenbaum's δ in the earlier literature

they satisfy any simple algebraic relations. On the question of the limits as d tends to ∞ of α and δ , see [Eckmann and Wittwer, 1985] and § 2.4 of this thesis.

I will next evaluate α and δ to high precision for various d, in order to provide data for testing conjectures concerning these numbers.

2.1.2   Method

Calculating α directly from the definition is impractical because it would involve finding high iterates of f, which are subject to accumulation of roundoff error, making it difficult to locate the bifurcation values μ_j accurately. (See, however, § A.)

Instead, as described in the introduction, we define an operator T, acting on functions $g : \mathbb{R} \to \mathbb{R}$, by

$$(Tg)(x) = g \circ g(g(1)x)/g(1).$$

If we find an even real analytic function invariant under T, with $g(0) = 1$, then α is determined by $\alpha = 1/g(1)$.

The numerical method proceeds by approximating g by the form

$$g(x) = 1 + \sum_{i=1}^{n} g_i |x|^{di}.$$

An approximate fixed point of T can then be found either by formal power series methods (see Chapter 7) or by a collocation method. I will use the latter approach

first, and then confirm my results with power series methods. In a collocation method, we require that $Tg = g$ be satisfied at n points x_j in the interval $(0, 1]$, and we then solve the resulting n nonlinear equations by an n-dimensional Newton-Raphson iteration [Newton, 1687; Raphson, 1690; Press, Flannery, Teukolsky and Vetterling, 1986]. Thus one requires (for $j = 1, 2, \ldots, n$) that

$$\left(1 + \sum_{i=1}^{n} g_i\right)\left(1 + \sum_{i=1}^{n} g_i |x_j|^{di}\right) - 1 - \sum_{k=1}^{n} g_k \left|1 + \sum_{i=1}^{n} g_i \left|(1 + \sum_{l=1}^{n} g_l)x_j\right|^{di}\right|^{dk} = 0.$$

If we call the left side of this equation f_j, the Newton iteration requires the inversion of the Jacobian matrix $\partial f_j / \partial g_i$. This is the major part of the computational task. For the smaller values of d, it was found that the initial approximation to the coefficients g_i was not critical, but for the larger values of d some trial and error was necessary before convergence was obtained. For d greater than 12, all initial approximations tried produced divergence of the Newton iteration. However, it is known that a solution does exist for all d.

Feigenbaum has shown [Feigenbaum, 1979] that the constant δ is the largest eigenvalue of the local linearization of T about the fixed point function g found above. A simple calculation shows that this operator DT (for fixed α) is given by

$$(DTf)(x) = \alpha g' \circ g(x/\alpha)f(x/\alpha) + \alpha f \circ g(x/\alpha).$$

Once the approximate fixed point g has been found, one may construct a finite-dimensional matrix approximating DT by a method similar to that used above. That is, one evaluates the right side of the above equation at the n nodes x_i. The largest eigenvalue of the matrix can then easily be found by the power method [Golub and Van Loan, 1989].

2.1.3 Results

I implemented the above scheme with arbitrary-precision floating point arithmetic using the methods described by Brent [Brent, 1978]. The choice of the nodes x_i was found to be not critical, the spacing $x_i = (i/n)^{1/d}$ producing the most stable results.

It is observed that the coefficients g_i decrease rapidly in magnitude; for example for $d = 2$, $|g_i|$ is about 10^{-i}. This gives a guide to the value of n needed; since α is $1/g(1)$, we must set n about equal to the number of decimals desired for α, and preferably greater. I first found α and δ for $d = 2, 3, \ldots, 12$ with $n = 75$ and a

$\alpha(2)$	-2.50290787509589282228390287321821578638127137672714997
	733619205677923546317959020670329964974643383412959523 2
$\delta(2)$	4.66920160910299067185320382046620161725818557747576863
	274565134300413433021131473713 87
$\alpha(3)$	-1.92769096384764084494999435296631905189265896703673262
	0743579672740866774900 09
$\delta(3)$	5.96796870377745104099419301997967232351260291982742394
	839317 20
$\alpha(4)$	-1.69030297140524485334378015032416134822827805970956196
	66824232 63
$\delta(4)$	7.28468621707334336430893056799555306947804661979979065 9
	072 12
$\alpha(5)$	-1.55577125019651840213297862965748441019232289917422932 9
$\delta(5)$	8.34949913206696352110974740181123558325747 6
$\alpha(6)$	-1.46774245031990094445383431510897374636879712939 67
$\delta(6)$	9.29624683277137008283447656636745755030668875 6
$\alpha(7)$	-1.40511078831683179942567128926679825719406 757
$\delta(7)$	10.22215952883488165524180132934 744
$\alpha(8)$	-1.35801727913805034548737633310626140065 806
$\delta(8)$	10.94862426594159042553420790071223 4803
$\alpha(9)$	-1.32118575980525276678233264501112163 344
$\delta(9)$	11.76833363955408532268157 502
$\alpha(10)$	-1.29151686726234456962559234290148 3728
$\delta(10)$	12.34140904534929383969763042 3331
$\alpha(11)$	-1.26706140790247246329005973368136 8673
$\delta(11)$	13.07654580565116239270 558
$\alpha(12)$	-1.24652775172074929543980658725193 12
$\delta(12)$	13.53507566617027649570056335 38

Table 2.3: Precise feigenvalues

working precision of 150 decimal places. I then repeated all calculations with $n =$ 100 and a working precision of 200 decimal places. The results given in Table 2.3 show as many digits as agree between the two calculations. Thus, it is probable that all listed digits are correct.

2.2   Algebraic relations

After computing the results of the previous section, I decided to check the feigen-values by re-implementing the computation to use formal power series methods (see Chapter 7) and the new mpfun multiple-precision software by David Bailey [Bailey, 1993*b*; Bailey, 1993*a*]. Running the calculation at a precision level of 576

$\alpha(2) = -2.$
5029078750958928222839028732182157863812713767271499773361920
5677923546317959020670329964974643383412959523186999585472394
2182377785445179272863314993372578112163594879503744781260997
3805986712397117373289276654044010306698313834600094139322364
4906578899512205843172507873377463087853424285351988587500042
3582469187408204281700901714823051821621632597520264070154734
3691632648595671163080236722434663921880833699674571235704898
7097355573665606299802070960232361775874092091791886260056503
7392462501491789236269217819393491578638085655372062878991364
7510711309308829555305 28

$\delta(2) = 4.$
6692016091029906718532038204662016172581855774757686327456513
4300413433021131473713868974402394801381716598485518981513440
8627142027932522312442988890890859944935463236713411532481714
2199474556443658237932020095610583305754586176522220703854106
4674949428498145339172620056875566595233987560382563722564800
4095107128389061184470277585428541980112212097941214729746823
4972701549082591194207813340879396740170788771896753500248657
2519776417775619778479591234753500973758935178262365904735075
2485156705948080910948188617490903004106492043004698687299477
0129187745138984920982 34382

reason, the 'theorem' may not be true. In any case, this computation virtually rules out the possibility that α and δ could be algebraic with small degree, in particular, rational with small denominators. Similar negative results were obtained for all other constants ($\alpha(4), \delta(4)$ etc.) that I tried. I also computed the frequencies of partial quotients in the continued fractions of these feigenvalues, without finding any significant deviation from the expected Gauss distribution. I know of no other work on the number-theoretic nature of the Feigenbaum constants.

2.3 Corrections to scaling in real maps

In this section I will discuss the generic aspects of scaling in period-doubling sequences in families of maps of the real line possessing non-integer degree. I show that the scaling behaviour in both the orbital and parameter spaces is governed by the same sequence of eigenvalues of the linearized renormalization operator. These eigenvalues are smooth functions of the degree of the maximum of the map.

2.3.1 Introduction

As described in the introduction, Feigenbaum [Feigenbaum, 1978; Feigenbaum, 1979] showed that period-doubling sequences in families of maps of the real line of the form

$$x \rightarrowtail f_\lambda(x) = \lambda - |x|^d; \qquad d = 2, 4$$

have the asymptotic behaviour

$$\Delta\lambda_k \equiv \lambda_{k+1} - \lambda_k \quad \sim \quad a_0[\delta(d)]^{-k} \qquad (2.3)$$

$$\phi_k \equiv f_{\lambda_k}^{<2^{k-1}>}(0) \quad \sim \quad b_0[\alpha(d)]^{-k} \qquad (2.4)$$

as $k \to \infty$. Here λ_k is the smallest parameter value at which f_λ possesses a superstable 2^k cycle, superscript $^{<n>}$ indicates n-fold composition, and a_0 and b_0 are constants. (a_0 and b_0 are of course dependent on d, but as I am not interested in their value, I suppress the dependence from the notation.) α and δ were claimed to be *universal*, that is, dependent on d only. Additionally, Feigenbaum gave an argument ([Feigenbaum, 1978], section 5) that the rate of convergence of $\alpha_k \equiv \phi_k/\phi_{k+1}$ to its limit is also δ . This amounts to claiming that equation (2.4) is the first term in an expansion

$$\phi_k \sim b_0[\alpha(d)]^{-k} + b_1[\alpha(d)\delta(d)]^{-k} + \cdots .$$

However, Feigenbaum had evidence was that this is true for $d = 2$ only. The present study arose out of a desire to understand this discrepancy, and more generally, the dependence on d of the spectrum of feigenvalues. The case $d = 2$ has been studied by Mao and Hu [Jian-min Mao and Hu, 1987], Liu and Young [Liu and Young, 1987] and by Reick [Reick, 1992]. Here I present results of a numerical study of the general d problem, and a plausibility argument to justify the assumptions used the numerical study.

2.3.2 Numerical results

In order to generalize the concept of Feigenbaum scaling, I postulate the following forms for the behaviour of $\Delta\lambda_k$ and ϕ_k:

$$\Delta\lambda_k = \sum_{i=0}^{\infty} \frac{a_i}{\delta_i^k} \tag{2.5}$$

$$\phi_k = \sum_{i=0}^{\infty} \frac{b_i}{\alpha_i^k} \tag{2.6}$$

where a_i and b_i are constants (the *amplitudes*), and $|\delta_i| < |\delta_j|$, $|\alpha_i| < |\alpha_j|$ if $i < j$.

To ascertain the validity of these expansions, I accurately computed from four to eight of the quantities δ_i and α_i for about 50 values of d between 1 and 10. Here δ_i and α_i are the constants to be determined, with $\delta_0 = \delta$ and $\alpha_0 = \alpha$. The exponents are named in order of increasing magnitude. It is necessary to compute typically 15 to 20 superstable parameter values λ_k, for which I used the standard Newton-Raphson iteration as described in Appendix A. The use of high precision arithmetic (50 to 100 decimal places) is essential, since up to 2^{20} iterations of f_λ are being computed, and higher precision is needed for larger d. In general, it was found that if, at a given precision, convergence of the Newton iteration was obtained at all, then the parameter value to which the iteration converged was correct. All results were confirmed by re-computation at higher precision levels than used for the quoted values.

I then computed the exponents by a method similar to that described by Mao and Hu [Jian-min Mao and Hu, 1987], again using high precision arithmetic. I illustrate this by the case of the δ scaling exponents, the case of α being exactly analogous. The problem reduces to solving a linear system and a polynomial equation, as follows. Assuming that N scaling exponents are desired, 2N successive values of $\Delta\lambda$ are required. Let us call these $\Delta\lambda_i, \ldots, \Delta\lambda_{i+2N-1}$. Eliminating the

constants a_i from the definition of the scaling exponents results in

$$
\begin{bmatrix}
-\Delta\lambda_{i+1} & +\Delta\lambda_{i+2} & \ldots & (-1)^N\Delta\lambda_{i+N} \\
-\Delta\lambda_{i+2} & +\Delta\lambda_{i+3} & \ldots & (-1)^N\Delta\lambda_{i+N+1} \\
\vdots & \vdots & \vdots & \vdots \\
-\Delta\lambda_{i+N} & +\Delta\lambda_{i+N+1} & \ldots & (-1)^N\Delta\lambda_{i+2N-1}
\end{bmatrix}
\begin{bmatrix}
t_1 \\ t_2 \\ \vdots \\ t_N
\end{bmatrix}
=
\begin{bmatrix}
-\Delta\lambda_i \\ -\Delta\lambda_{i+1} \\ \vdots \\ -\Delta\lambda_{i+N-1}
\end{bmatrix}
$$

where $t_1 = \sum_{i=1}^N \delta_i$, $t_2 = \sum_{i\neq j}^N \delta_i\delta_j$ etc. This equation is solved for $\{t_1, t_2, \ldots, t_N\}$, and the roots of the polynomial

$$
x^N - x^{N-1}t_1 + \cdots + (-1)^N t_N
$$

are found. The N roots are the desired scaling exponents δ_i. This calculation is identical to the construction of a Padé approximant to the analytic function $f(x) = \sum_{i=1}^\infty \Delta\lambda_i x^i$, and the subsequent determination of the poles δ_i of $f(x)$. This point of view suggests that in the case of integer d, where δ is known to very high accuracy (as in § 2.1), elimination of the pole δ by multiplication of $f(x)$ by $1 - x/\delta$ might achieve greater accuracy in the calculation of the higher poles δ_i. However, I have not found a sufficient increase in precision to make this procedure worthwhile. However, the Padé approximant point of view will prove useful in § 4.4.

A selection of numerical values are shown in Table 2.5, and the full results are shown in Figures 2.1 to 2.4. In all figures the exponents are numbered in order of increasing magnitude. In these tables, the word 'origin' refers to the explanation of the numerical values in terms of the eigenvalues μ_i, and the gaps are unobserved scaling exponents. For reasons that will become apparent, it is most useful to consider δ_i/δ and $\alpha_i/\alpha, i = 1, 2, 3, \ldots$, rather than the exponents themselves. Only as many decimal places as I believe correct are quoted.

2.3.3 Theory

All the above results can be understood by an extension of an argument of Feigenbaum [Feigenbaum, 1980a]. I define the usual doubling operator T, by

$$
(Tf)(x) = \alpha f \circ f(\alpha^{-1}x),
$$

where α is the (assumed known) value appropriate to the degree d of the maximum of f. A Taylor expansion in λ of f_λ about λ_∞ gives

$$
\begin{aligned}
f_\lambda &= f_{\lambda_\infty} + (\lambda - \lambda_\infty)\frac{\partial f_\lambda}{\partial \lambda} + \cdots \\
&= g_0(x) + (\lambda - \lambda_\infty)h_0(x) + \cdots.
\end{aligned}
$$

d = 2		
$\delta = 4.6692, \alpha = -2.5029$		
δ_i/δ	α_i/α	origin
4.6692	4.6692	δ
	6.2645	α^2
−8.0872	−8.0872	μ_1^{-1}
−17.4499	−17.4499	μ_2^{-1}
21.8014	21.8	δ^2
	29.9	$\delta\alpha^2$
−37.76		$\delta\mu_1^{-1}$
64		μ_1^{-2}
−84		$\delta\mu_2^{-1}$
$\mu_1^{-1} = -8.087165, \mu_2^{-1} = -17.4499$		

d = 4		
$\delta = 7.2846, \alpha = -1.6903$		
δ_i/δ	α_i/α	origin
3.4266	3.4266	μ_1^{-1}
−3.9114	−3.9114	μ_2^{-1}
−6.4283	−6.4283	μ_3^{-1}
7.2846	7.3	δ
−13		$\delta_1\delta_2$
$\mu_1^{-1} = 3.4266, \mu_2^{-1} = -3.9114$ $\mu_3^{-1} = -6.4282$		

d = 6		
$\delta = 9.2962, \alpha = -1.4677$		
δ_i/δ	α_i/α	origin
2.6949	2.6949	μ_1^{-1}
−2.9242	−2.9242	μ_2^{-1}
−4.7499	−4.7495	μ_3^{-1}
4.8100	4.8112	μ_4^{-1}
−6.8	−6.8	μ_5^{-1}
7	7.3	μ_1^{-2}
$\mu_1^{-1} = 2.6949, \mu_2^{-1} = -2.9242$ $\mu_3^{-1} = 4.7499, \mu_4^{-1} = 4.8100$ $\mu_5^{-1} = -6.901$		

d = 8		
$\delta = 10.948, \alpha = -1.3580$		
δ_i/δ	α_i/α	origin
2.3285	2.3285	μ_1^{-1}
−2.4620	−2.4620	μ_2^{-1}
3.6309	3.63	μ_3^{-1}
−3.94	−3.9	μ_4^{-1}
−4.992	−5	μ_5^{-1}
5.463	5.4	μ_1^{-2}
μ_i not computed		

Table 2.5: The dominant scaling exponents for even integer d

g_k and h_k may now be defined by

$$(T^k f_\lambda)(x) = g_k(x) + (\lambda - \lambda_\infty)h_k(x) + \cdots,$$

and it is known from the work of Feigenbaum [Feigenbaum, 1980a] that as $k \to \infty$

$$g_k(x) \quad \to \quad g(x)$$
$$h_k(x) \quad \to \quad (DT_g)(h_{k-1})(x).$$

Here $g(x)$ satisfies $Tg = g$, DT_g is the linearization of T about g, and $\alpha^{-1} = g(0)$. It is known that Feigenbaum's δ is the dominant eigenvalue of T_g, and governs the approach of λ_k to λ_∞. I will now compute the effect of the subdominant eigenvalues.

Let us expand $h_0(x)$ in eigenvectors $e_j(x)$, $(j = 1, 2, \ldots)$ of DT_g. Letting $DTe_j =$
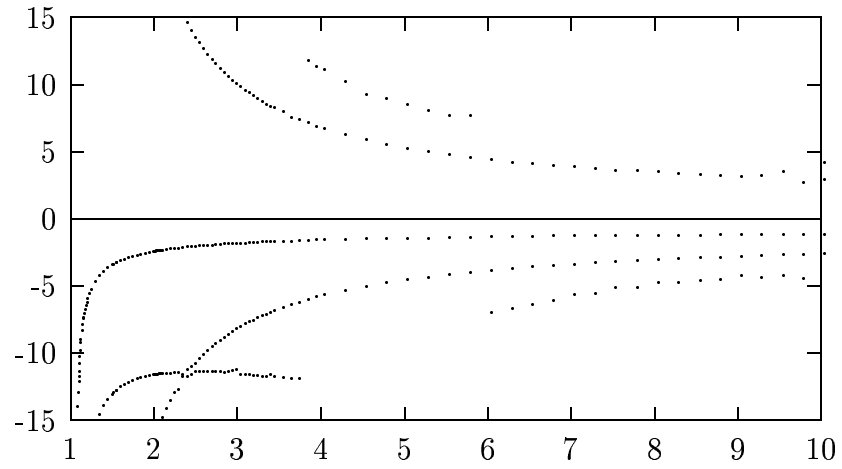
Figure 2.1: The scaling exponents α_i vs. $d, i = 0, 1, 2, 3$
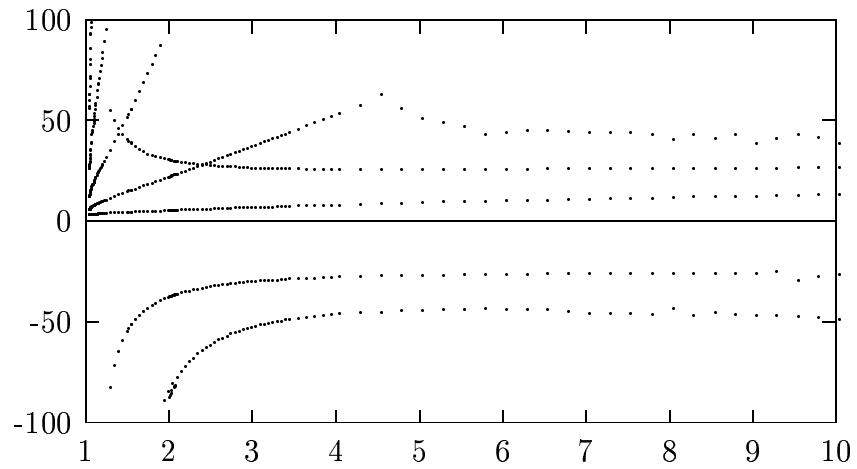
Figure 2.2: The scaling exponents δ_i vs. $d, i = 0, 1, 2, 3, 4$

$\mu_j e_j$ with $e_j(0) = 1$ and $\mu_1 = \delta$ gives

$$h_0(x) = \sum_{j=1}^{\infty} a_j e_j(x)$$

$$h_k(x) = a_1 \delta^k \left[e_1(x) + \sum_{j=2}^{\infty} \frac{a_j}{a_1} \left(\frac{\mu_j}{\mu_1} \right)^k e_j(x) \right].$$
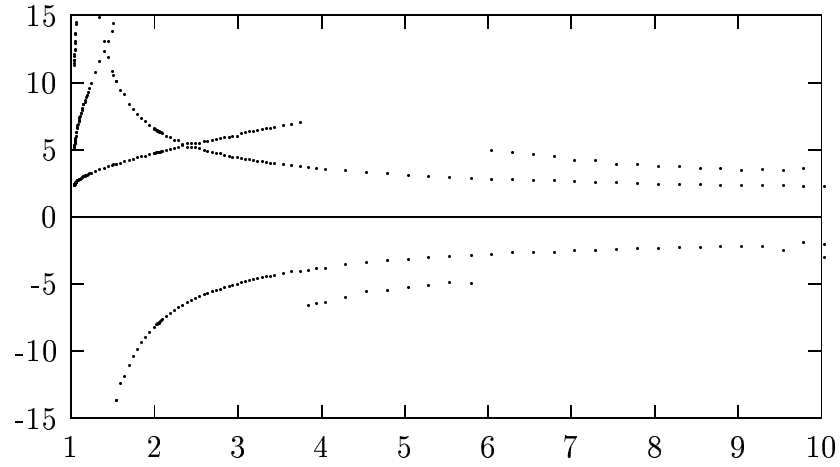
Figure 2.3: The ratios α_i/α vs. $d, i = 0, 1, 2, 3$
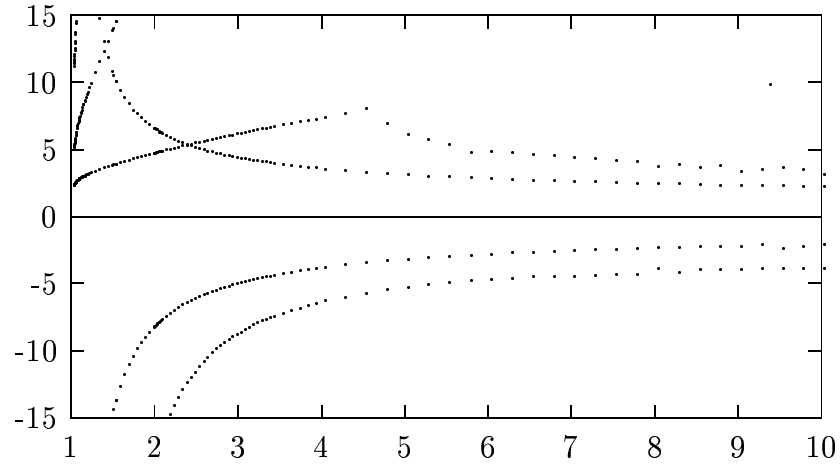
Figure 2.4: The ratios δ_i/δ vs. $d, i = 0, 1, 2, 3, 4$

Thus as $k \rightarrow \infty$,

$$\alpha^k f_{\lambda_k}^{<2^k>}(\alpha^{-k}x) \sim g(x) + (\lambda_k - \lambda_\infty)a_1\delta^k \left[e_1(x) + \sum_{j=2}^{\infty} \frac{a_j}{a_1} \left(\frac{\mu_j}{\mu_1} \right)^k e_j(x) \right],$$

and so

$$\lambda_\infty - \lambda_k \sim a_1^{-1}\delta^{-k} \left[1 + \sum_{j=2}^{\infty} \frac{a_j}{a_1} \left(\frac{\mu_j}{\mu_1} \right)^k \right]^{-1}.$$

Inverting the sum in square brackets shows that

$$\lambda_\infty - \lambda_k \sim \sum_{j=1}^\infty b_j \delta_j^k,$$

where each δ_j is of the form $\delta \prod_{k=1}^\infty \mu_k^{-m_k}$, ($m_k$ being non-negative integers), and $\{b_j\}$ are constants which are complicated functions of $\{a_j\}$.

This completes the computation of δ_j in terms of $\{\mu_j\}$. It is clear that the behaviour can be pictured as an approach of f_{λ_k} to f_{λ_∞}, which for an arbitrarily chosen parameterization of f_λ, will not be along the eigendirection corresponding to δ. Nevertheless, the more general approach is still governed by the largest eigenvalues of DT_g.

I now perform a similar calculation for α_j. Applying the doubling operator 2^{k-1} times to f_{λ_k} gives

$$\alpha^k f_{\lambda_k}^{<2^{k-1}>}(\alpha^{-(k-1)}x) \sim g(x) + (\lambda_k - \lambda_\infty)a_1\delta^{k-1}\left[e_1(x) + \sum_{j=1}^\infty \frac{a_j}{a_1}\left(\frac{\mu_j}{\mu_1}\right)^{k-1}e_j(x)\right].$$

Thus

$$\alpha^k \phi_k \sim 1 + (\lambda_k - \lambda_\infty)a_1^{-1}\delta^k\left[1 + \sum_{j=1}^\infty \frac{a_j}{a_1}\left(\frac{\mu_j}{\mu_1}\right)^k\right]^{-1}$$

(recalling that $\phi_k \equiv f_{\lambda_k}^{<2^{k-1}>}(0)$). Now performing an inversion as above gives

$$\phi_k \sim \alpha^{-k}\left[1 + \sum_{j=1}^\infty c_j\delta_j^k\right],$$

where c_j are constants and δ_j are the same scaling exponents as above. The important result is that the one set of eigenvalues μ_j governs both the parameter *and* the orbital scaling. The latter simply have an extra factor of α.

There is a curious phenomenon in the case $d = 2$. The largest eigenvalue less than one takes the value exactly α^{-2}. In other words, there is an eigenvalue crossing at $d = 2$. As expected, the effect of the eigenvalue is observed in the orbital scaling, but surprisingly, it is *not observed* in the parameter scaling. I conjecture that this is because the corresponding amplitude a_i is zero (or very small). This effect nevertheless deserves more detailed study. This appears to be the only case where $\mu_j = \alpha^{-k}$ for any j and k, though there are close approaches at $d = 4, 6$ and 8. These close exponents make the numerical calculation difficult, and this explains the oscillations seen in the graphs around $d = 8$ to 10. I believe

that a more accurate calculation would show that all the exponents are monotonic functions of the degree of the maximum d.

The eigenvalues μ_j can be computed independently for integer d via a numerical approximation to DT_g, as described in § 2.1. This can be achieved this by first computing a finite-dimensional approximation to $g(x)$ in the form $g(x) = \sum_{i=0}^{N} g_i x^{di}$. This is found by solving $Tg - g = 0$ by a Newton iteration in the space of Taylor coefficients g_i. Similarly DT_g is represented by a $N \times N$ matrix, and all its eigenvalues computed. The fact that an infinite-dimensional problem has been truncated to finite dimension means that some extra eigenvalues are introduced, but the relevant eigenvalues are usually readily identified by comparison with the bifurcation data. It is known that α^{-k} is an eigenvalue for integer k [Feigenbaum, 1979], and Feigenbaum conjectured that these are all the eigenvalues of modulus less than unity. It is clear now that this is not the case. The eigenfunctions corresponding to even integer k are even, so that for odd integer d, our expansions cannot represent the eigenfunctions, and the eigenvalues are not found. However, this is not a difficulty as these eigenvalues are not relevant to the bifurcation data.

I list a few of the largest eigenvalues μ_i for even integer d at the bottom of the tables in Table 2.5. Observe that eigenvalues *not* of the form α^{-k} are also present. This then is the explanation of the discrepancy between the behaviour at d = 2 and d = 4 mentioned in § 2.3.1.

Of interest is the question of the limits d → 1 and d → ∞ of the scaling spectrum. Collet et al. [Collet, Eckmann and Lanford, 1980] have shown that δ → 2 as d → 1. Eckmann and Wittwer [Eckmann and Wittwer, 1985] and van der Weele et al. [van der Weele, Capel and Kluiving, 1986; van der Weele, 1987] have shown that δ →≈ 30 as d → ∞. I will examine this limit in detail in § 2.4. It is also known [Delbourgo, 1992] that α → −∞ as d → 1, and that α → −1 as d → ∞. Unfortunately the perturbation method (about d = 1) used by Collet et al. gives only the behaviour of the largest eigenvalue δ. It will be observed from the graphs that all the eigenvalues appear to be smooth functions of d, despite the fact that there are crossings in the exponents. Notice also that powers of δ appear in the scaling spectrum, and that these dominate the behaviour for d close to one. These powers of δ can be seen as the rapidly increasing functions near d = 0 in Figure 2.2. Conversely, the behaviour for large d is governed by a number of eigenvalues which approach each other in magnitude. In fact, these eigenvalues appear to occur in pairs of approximately equal magnitude but opposite

sign. The numerical calculations suggest that the following limits exist, and take the approximate numerical values given:

$$\lim_{d\to\infty} \delta_1/\delta \approx -\lim_{d\to\infty} \delta_2/\delta \approx 2$$
$$\lim_{d\to\infty} \alpha_1/\alpha \approx -\lim_{d\to\infty} \alpha_2/\alpha \approx 2.$$

I have thus shown that all observable aspects of the period-doubling sequence in families of maps of the form $\lambda - |x|^d$ are governed by the eigenvalues of the linearized period-doubling operator. This generalizes the original Feigenbaum scaling law. The same is no doubt true in the case of complex maps $\lambda - z^m$ (with integer $m > 0$) in the case of n-tupling ($n = 2, 3, 4, \ldots$). In § 3.2.2 I describe some additional calculations for the cases of period-doubling and period-tripling and $m = 2, 3, \ldots, 8$ which confirm this.

2.4 Analytic solutions of the Feigenbaum functional equation

I began the study described in this section by trying to rigorize the results of [Thompson and McGuire, 1988] on essentially singular solutions of the Feigenbaum functional equation. In this paper, Thompson and McGuire, by means of an inspired ansatz, found an approximate solution of the Feigenbaum functional equation having an essential singularity. I was intrigued by this work, and wanted to see whether the number of terms in the series part of the solution could be increased indefinitely. This turned out to be difficult, as the series is divergent, but I eventually succeeded.

It is well to begin to noting that the Feigenbaum nonlinear functional equation arising in the scaling theory of maps of the real line can be related to the classical Schröder and Abel linear functional equations. These functional equations are discussed in Chapter 7. The link will be used here to obtain information about the analytic solutions, and in particular the essentially singular solution, of the Feigenbaum equation, providing an alternative description of the latter to that of Eckmann and Wittwer. I obtain a very accurate numerical approximation to this singular solution, using special techniques to handle the divergent series. This accuracy is a substantial improvement on previous estimates of the solution, and allows a very precise determination of the associated asymptotic feigenvalues.

In § 4.3, the solutions of the Feigenbaum-Kadanoff-Shenker equation for scaling in circle maps are solved by a similar method.

2.4.1   Introduction

As described in § 1.2, Feigenbaum's discovery [Feigenbaum, 1978] of scaling properties of the iterates of parameterized families of unimodal maps $x \rightarrowtail \phi_\mu(x)$ of intervals of the real line has suggested an explanation in terms of the Feigenbaum functional equation

$$f \circ f(\gamma x) = \gamma f(x), \quad \gamma > 0 \qquad \text{(FFE)}$$

and its analytic solutions. Note that here γ is the reciprocal of the constant α used in § 2.1. Such solutions and the values of the constant γ depend on the behaviour of the solution function at its unique extremum. Let us scale the interval on which the solution f is defined so that f maps the interval $[0,1]$ into $[0,1]$.

In the original version of the equation ([Feigenbaum, 1979]) the right hand side of (FFE) had a negative sign and f had a maximum; I will instead use a form first introduced by McGuire and Thompson [McGuire and Thompson, 1982]. This is simply obtained from the Feigenbaum form by conjugation with $x \rightarrowtail 1 - x$.

It is then found that solutions f are such that the graph of f decreases monotonically from $f(0) = 1$ to its minimum $f(b) = 0$ and then rises monotonically to $f(1) = \gamma$. I shall also impose the condition that the solution should have non-zero derivative at $x = 0$. Such solutions are called *strongly unimodal*. Solutions without a singularity (either essential or algebraic) at b will be called *regular*.

Now if f is regular at its zero b, it must behave like $c(b-x)^N$ for some positive even integer N in the neighbourhood of b. I shall call such a solution regular of order N and denote it f_N; its existence is well established; for example, [Koch et al., 1994] prove the existence for $N = 2$; [Campanino et al., 1982] and [Epstein, 1986] prove the existence for $2 \leqslant N \leqslant 14$. Also see § 2.1 for accurate values of the associated constants. It will become apparent that the key to understanding the limiting behaviour of γ as $N \to \infty$ is the study of non-regular solutions of the Feigenbaum functional equation.

We next need to consider the invariance of the Feigenbaum functional equation under the *power conjugacy* transformation

$$f_{N,\lambda}(x) = \left[f_N(x^{1/\lambda}) \right]^\lambda, \qquad \lambda > 0.$$
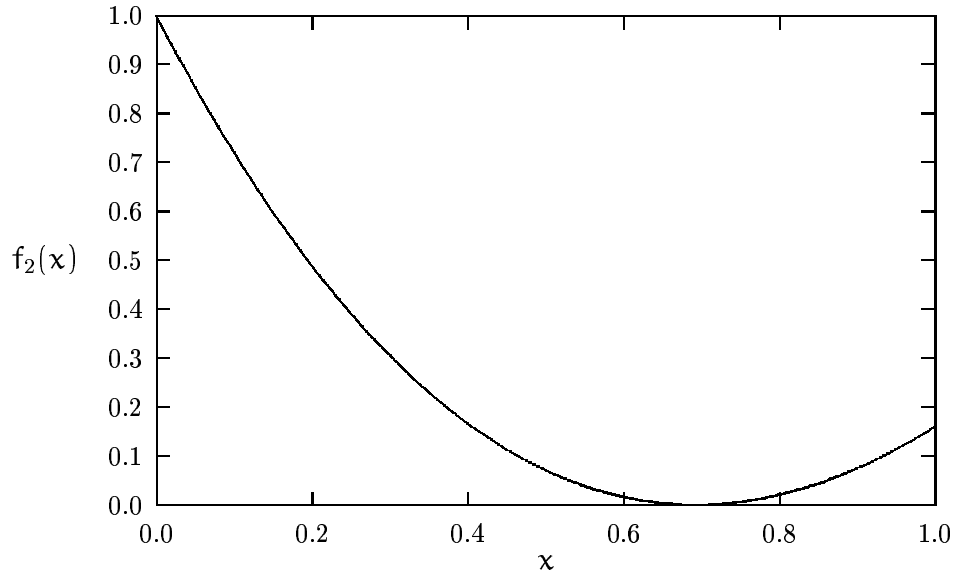
Figure 2.5: The solution of FFE for $N = 2$

It carries f_N into a $f_{N,\lambda}$ which also satisfies (FFE) with associated constant γ^λ and minimum at b^λ. At this minimum it behaves like $c_\lambda(b^\lambda - x)^{N\lambda}$. If $N\lambda$ is an even integer then $f_{N,\lambda}$ is again a regular solution; otherwise it has an algebraic-logarithmic singularity at b^λ. Hence in every conjugacy class of a regular f_N there are infinitely many regular solutions, but at most one of these will be strongly unimodal. There is no *a priori* reason why the strongly unimodal solution should be regular but indeed for every even integer $N > 0$ there seems to exist a unique regular unimodal solution which has a negative derivative at zero. This is clear from existing literature and is confirmed by the computational evidence of the present section.

If it exists, I call this the *principal* solution of order N, and the symbol f_N will be used for this particular solution. See Figure 2.5 for a plot of the principal solution f_2.

Apart from the f_N there is another possibility to be considered: a conjugacy class of solutions of (FFE) characterized by the property that members of the family tend to zero at b more rapidly than any positive power of $|b - x|$. In an elaborate computer-assisted study, Eckmann and Wittwer [Eckmann and Wittwer, 1985] proved the existence of a singular analytic solution. Epstein has also proved the existence of this solution in Appendix 2 of [Epstein, 1986]. My aim in the present section is to describe a method of accurately constructing this solution f_∞.

Now since all power conjugacies of this solution f_∞ have the same kind of essential algebraic-logarithmic singularity at their minimum b we cannot distinguish a representative of the class through its behaviour at b. However, there is at most one strongly unimodal member of the family and this particular representative will be called the singular solution and denoted by f_∞. The exact form of the essential singularity will be given later.

We must next consider the analytic nature of the solution to the left and right of the singular point b. This question does not arise with the principal solutions (including the original Feigenbaum $N = 2$ solution) where the analytic continuation of the left-hand solution is the right-hand solution. However, from the point of view of the singular solution, the functional equation has a curious anomaly: the solution function to the left and to the right of b involves two distinct analytic functions which are not necessarily analytic continuations of one another, and yet the two functions are denoted by the same symbol f in the equation. Since our main interest here is in the singular solution, it is necessary to separate the rôles of the two functions. This is achieved by considering an iterated *one-sided* form of (FFE). This equation then involves only the left hand interval [0,b]. It is

$$f \circ f(\gamma f(\gamma x)) = \gamma^2 f(x). \qquad \text{(FFE*)}$$

From the solution of (FFE*) we can then easily obtain a solution of the original (FFE) for the whole interval [0,1]. Another advantage of (FFE*) is that it leads more directly to a link with Schröder's and Abel's functional equations which will form the basis of the present method. In § 4.3 I will also discuss the Feigenbaum-Kadanoff-Shenker equation

$$g \circ g(\epsilon^2 x) = \epsilon g(x), \quad \epsilon < 0. \qquad \text{(FKS)}$$

2.4.2   The principal and singular solutions

I will next solve the Feigenbaum functional equation

$$f(x) = \gamma^{-1} f \circ f(\gamma x) \qquad \text{(FFE)}$$

for an analytic f which decreases monotonically from $f(0) = a > 0$ to $f(ab) = 0$ for some $b > 0$ and then increases monotonically to $f(a) = f(f(0)) = \gamma f(0) = \gamma a$. Note that in this formulation, $0 < \gamma < 1$. There is no restriction in generality if we assume that $a = 1$, for the transformation $f(x) = a\hat{f}(x/a)$ carries (FFE) into

$$\hat{f} \circ \hat{f}(\gamma y) = \gamma \hat{f}(y), \quad y = x/a,$$

that is, a Feigenbaum functional equation with the same γ and $\hat{f}(0) = 1$. After this is done, we have

$$f(0) = 1, \quad f(b) = 0, \quad f(1) = \gamma. \tag{2.7}$$

Note also that

$$f \circ f(\gamma) = \gamma^2, \quad f(\gamma b) = b, \tag{2.8}$$

obtained by setting $x = 1$ and $x = b$ respectively into (FFE) and observing (2.7).

Since the solutions may have a singularity at b, we have to be careful to label the solutions to the left and right of b differently. Let $v(x)$ be the restriction of f to $b < x \leqslant 1$ and f the restriction to $0 \leqslant x < b$; then (FFE) takes the form

$$f \circ f(\gamma x) = \gamma v(x), \quad \text{for} \quad b < x \leqslant 1 \tag{2.9}$$

and

$$v \circ f(\gamma x) = \gamma f(x), \quad \text{for} \quad 0 \leqslant x < b. \tag{2.10}$$

We may eliminate v from (2.9) and (2.10) by setting $x = f(\gamma y)$ for $0 \leqslant y < b$ (hence $b < x \leqslant 1$); then (FFE) becomes

$$f \circ f(\gamma f(\gamma y)) = \gamma^2 f(y), \quad \text{for} \quad 0 \leqslant y < b, \tag{FFE1}$$

an equation which only involves $f : [0, b) \rightarrow (0, 1]$. This iterated form of the equation is equivalent to the original (FFE), for if we solve (FFE*) and then define $v(x)$ through (2.9), then (2.10) is automatically satisfied and (FFE) is also solved. Of course if f is an analytic solution of (FFE*) then $v : (b, 1] \rightarrow (0, \gamma]$ is also analytic, but not necessarily an analytic continuation of f.

From (2.10) we get $v' \circ f(\gamma x)f'(\gamma x) = f'(x)$ and setting $x = 0$, and assuming $f'(0) \neq 0$, $f'(0) \neq -\infty$, we get $v'(1) = 1$. Similarly, from (2.9), $f' \circ f(\gamma x)f'(\gamma x) = v'(x)$ and hence with $x = 1$

$$f' \circ f(\gamma)f'(\gamma) = 1. \tag{2.11}$$

Any solution of (FFE*) which satisfies $f(0) = 1$ and hence (2.8) must satisfy (2.11). This provides a check for the accuracy of the calculations. In what follows I will not be concerned with v, but rather will concentrate on solutions of (FFE*).

To get a better picture of the solution, let us denote by f_k the restriction of f to the interval $I_k = [\gamma^{k+1}b, \gamma^k b)$, $k = 0, 1, 2, \ldots$. We know that γ has a value

about 0.03, so that the intervals accumulate rapidly at zero. Since $f(\gamma b) = b$, we have also $\gamma f(\gamma y) > \gamma b$, from which it follows that $f(\gamma y) = f_{k+1}(\gamma y) > b$ for $y \in I_k$. It follows immediately that we have $f(\gamma f(\gamma y)) = f_0(\gamma f_{k+1}(\gamma y))$ and thus $f \circ f_0(\gamma f_{k+1}(\gamma y)) = \gamma^2 f(y) < \gamma^2 = f \circ f(\gamma)$. From this we see that $f_0(\gamma f_{k+1}(\gamma y)) > \gamma b$ and so (FFE*) becomes

$$f_0 \circ f_0(\gamma f_{k+1}(\gamma y)) = \gamma^2 f_k(y) \quad \text{for} \ \ y \in I_k. \tag{FFE2}$$

In particular

$$f_0 \circ f_0(\gamma f_1(\gamma y)) = \gamma^2 f_0(y) \quad \text{for} \ \ y \in I_0 = [\gamma b, b). \tag{FFE3}$$

The problem then is to determine f_0 so that f_1 are the analytic continuation of f_0. Once this is achieved, (FFE2) merely becomes an equation for the analytic continuation of f_0 over the whole interval $[0, b)$. Henceforth I will confine my attention to (FFE3).

Since $f(\gamma b) = b$ and f is supposed to be analytic at γb, we may set

$$f(\gamma b + \gamma t) = b + b_1 t + b_2 t^2 + \cdots \quad \text{with} \ \ b_1 < 0$$

and assume convergence of the series for some $|t| < r$, $r > 0$. Next set $\psi(x) = f(\gamma x)$, $h(x) = \psi \circ \psi(x)$; then (writing f for both f_0 and f_1) (FFE3) takes the form

$$f \circ h(x) = \gamma^2 f(x). \tag{S}$$

This is Schröder's functional equation for the unknown function f. We may solve it in a neighbourhood of a fixed point of h by the formal power series algorithm described in Chapter 7. Such a fixed point is b; indeed

$$h(b) = \psi \circ \psi(b) = \psi \circ f(\gamma b) = \psi(b) = f(\gamma b) = b.$$

For convenience I transform the fixed point to $u = 0$ by defining

$$h^*(u) = b - h(b - u) \text{and} f^*(u) = f(b - u);$$

then $h^*(0) = 0$ and

$$f^* \circ h^*(u) = \gamma^2 f^*(u). \tag{S*}$$

To solve (S*) for f^* we must know the Taylor expansion of $h^*(u)$. Now

$$
\begin{aligned}
\psi(b + t) \ \ &= f(\gamma b + \gamma t) = b + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \\
h(b - u) \ \ &= \psi(b - b_1 u + b_2 u^2 - b_3 u^3 + \cdots) \\
&= b + b_1(-b_1 u + b_2 u^2 - \cdots) + b_2(-b_1 u + b_2 u^2 - \cdots)^2 + \cdots
\end{aligned}
$$

hence

$$h^*(u) = b - h(b - u) = b_1^2 u + a_2 u^2 + a_3 u^3 + \cdots \qquad (2.12)$$

where

$$a_2 = -b_1 b_2(1 + b_1), \ a_3 = b_1 b_3(1 + b_1^2) + 2b_1 b_2^2, \ \text{etc.}$$

Suppose that $b_1^2 \neq 1$ in (2.12), then Schröder's equation (S*) has a convergent solution of the form

$$f^*(u) = cu^N(1 + c_1 u + c_2 u^2 + \cdots) \qquad (2.13)$$

with arbitrary $c > 0$, $N > 0$ and the coefficients c_k obtained by formal substitution of (2.12) into (S*). In particular the coefficient of u^N gives $b_1^{2N} = \gamma^2$,

$$\gamma = |b_1|^N, \ b_1 = -\gamma^{1/N}.$$

The constant c in (2.13) is determined from the condition $f^*(b - \gamma b) = b$, while b is adjusted so that $f(0) = 1$, $f^*(b) = 1$. In the case of the principal solution of order $N = 2$ we have $b_1 = -\sqrt{\gamma}$.

Substituting $u = b(1 - \gamma) - \gamma t$ into (2.13) we can transform back to centre γb and obtain

$$f(\gamma b + \gamma t) = b + b_1^* t + b_2^* t^2 + \cdots \qquad (2.14)$$

The coefficients b_k^* must be identical with the original coefficients b_k. This is the basis of the computational method described in the next section.

Suppose that $b_1^2 = 1$ in (2.12), that is

$$\gamma f'(\gamma b) = b_1 = -1. \qquad (2.15)$$

We then have

$$\psi(b + t) = f(\gamma b + \gamma t) = b - t + b_2 t^2 + b_3 t^3 + \cdots \qquad (2.16)$$

and $a_2 = 0$ in (2.12),

$$h^*(u) = u + a_3 u^3 + a_4 u^4 + \cdots \qquad (2.17)$$

with $a_3 = -2b_3 - 2b_2^2$, $a_4 = -b_2 b_3 - b_2^3$, etc. This characterizes the singular solution: the coefficient of u in the Taylor expansion of $h^*(u)$ (the so-called *multiplier*

of h^*) is one; that is, (2.15) is satisfied, and the coefficient of u^2 is zero. In such multiplier unity cases it is necessary to use Abel's functional equation instead of Schröder's (see Chapter 7). This is achieved by defining

$$A(u) = \frac{\log f^*(u)}{\log \gamma^2};$$ (2.18)

it satisfies

$$A \circ h^*(u) = A(u) + 1,$$ (A)

that is, Abel's functional equation for $h^*(u)$. The solution of this equation is not unique, even if we disregard the arbitrary additive constant; however there is a unique analytic solution (up to an additive constant) with an appropriate regular asymptotic behaviour when $u \to 0^+$. This is shown in [Szekeres, 1958]. The problem can now be expressed as follows: find $0 < \gamma < b < 1$ and an analytic $f : (0, b) \to (0, 1)$ such that $h^*(u)$ is defined through (2.16) and (2.17) and $A(u)$ through (2.18), then $A(u)$ is the regular solution of (A). We cannot drop the regularity requirement without conflicting with the monotonicity of the solution near b.

The form of the regular solution of (A) depends on the Taylor coefficients of $h^*(u)$ and in particular on the fact that the coefficient of u^2 in (2.17) is zero. Assuming that $b_3 \neq -b_2^2$, that is, $a_3 \neq 0$, $A(u)$ has the form (see Chapter 7)

$$A(u) = c_{-2}/u^2 + c_{-1}/u + c_0 \log u + c + c_1 u + c_2 u^2 + \cdots.$$ (2.19)

The coefficients c_k can be obtained by formal substitution of (2.17) and (2.18) into (A). In particular

$$c_{-2} = -\frac{1}{2a_3}, \qquad c_{-1} = \frac{a_4}{a_3^2}, \qquad c_0 = \frac{3}{2} - \frac{a_5}{a_3^2} + \frac{a_4^2}{a_3^3} \qquad \text{etc.}$$

For the complete algorithm I used for this procedure, see Chapter 7. The series (2.19) is not in general convergent; however, it can be an asymptotic expansion. The additive constant c is free; it is however determined by the boundary condition $f(\gamma b) = b$. Similarly γ is determined by the condition $f(f(\gamma)) = \gamma^2$. It follows from (2.18) that $f^*(u) = exp(A(u) \log \gamma^2)$, hence

$$f^*(x) =$$

$$exp\left[\log \gamma^2 (c_{-2}(b-x)^{-2} + c_{-1}(b-x)^{-1} + c_0 \log(b-x) + c + c_1(b-x) + \cdots)\right]$$

and

$$f^*(\gamma b + \gamma t) =$$

$$\exp\left[\log\gamma^2\left(\frac{c_{-2}}{(B-\gamma t)^2} + \frac{c_{-1}}{(B-\gamma t)} + c_0\log(B-\gamma t) + c + \sum_{k\geqslant 1}c_k(B-\gamma t)^k\right)\right],$$

where $B = b(1-\gamma)$, and c is determined from

$$b = \exp\left[\log\gamma^2(c_{-2}B^{-2} + c_{-1}B^{-1} + c_0\log B + c + c_1 B + c_2 B^2 + \cdots)\right].$$

Writing the expansion for f^* in the form

$$b - t + b_2^* t^2 + b_3^* t^3 + \cdots,$$

the task is then to determine the coefficients b_k so that $b_k^* = b_k$ holds for all k.

2.4.3   Computing the regular and singular solutions

I begin with the principal solution $f = f_N$, of even order N, including the original Feigenbaum function f_2. Given N, I start by fixing the number of terms, n, that I wish to retain in the expansion (2.4.2), and set tentative values for $0 < b < 1$, $-1 < b_1 < 0$, $b_2, \ldots, b_n$; we may set initially $b_i = 0$ for all $i \geqslant 2$, as long as b_1 is set reasonably close to its true value. Then I proceed to calculate: firstly the coefficients a_i in the expansion (2.12); secondly the coefficients c_i in (2.13) by formal substitution into (S*); thirdly c in (2.13) from

$$f^*(B) = cB^N(1 + c_1 B + c_2 B^2 + \cdots + c_n B^n) = b$$

where $B = b(1-\gamma)$ and $\gamma = |b_1|^N$, and, finally, the coefficients b_i^* in (2.14) by writing $u = B - \gamma t$ in (2.13), that is

$$c((B-\gamma t)^N + \cdots + c_n(B-\gamma t)^{N+n} \equiv b + b_1^* t + \cdots + b_n^* t^n \pmod{t^{n+1}}.$$

Using the Newton-Raphson method [Press et al., 1986] I solve for b_i from the implicit equations $b_i^* = b_i$, $i = 1, 2, \ldots, n$. The required partial derivatives were computed exactly using automatic differentiation techniques; see Appendix B for details of this. I also developed software for formal power series computations to facilitate the series manipulations needed, as described in Chapter 7. The desired accuracy is deemed to be obtained if repeating the calculation with a higher value of n does not produce any further change in the value of γ within an error of 10^{-10}.

N	n	γ	b	$-f'_N(0)$
2	14	0.1596284403	0.6928352170	0.4877073455
4	18	0.1225016054	0.5753037447	0.8987246525
6	23	0.1000232286	0.5267413888	1.1449080625
8	28	0.0864485340	0.4992100891	1.3121870867
10	36	0.0774460327	0.4812691791	1.4340790413
12	38	0.0710520444	0.4685776493	1.5271883443
14	46	0.0662791902	0.4590940595	1.6007980392
16	52	0.0625809693	0.4517230908	1.6605413729
18	62	0.0596312782	0.4458212181	1.7100498574
20	70	0.0572237129	0.4409842396	1.7517766788

Table 2.6: Properties of regular solutions of the FFE equation

The value of b is fixed by the normalization condition $f(0) = 1$; if the chosen value of b gives

$$f(0) = b + \sum_{i=1}^{n} b_i b^i = a \neq 1$$

then we simply replace b by b/a and b_i by $b_i a^{i-1}$, $i = 1, 2, \ldots, n$. This is an equivalent solution satisfying $f(0) = 1$. The method was used for $N = 2, 4, \ldots, 20$ and the results of Table 2.6 were obtained.

The last column shows that the method automatically produces the principal (strongly unimodal) solution; this is hardly surprising as the calculation tends to produce power series with the largest radius of convergence. The second column of the table lists the number of coefficients used; the higher the order N the more coefficients were needed to obtain 10 decimal place accuracy. There was no change in the character of the solution when Epstein's critical value $N = 14$ was raised to $N = 20$.

For $N = 2$ the method gave $b_1 = -\sqrt{\gamma} = -0.3995352805231$, the well known Feigenbaum value. Table 2.7 gives the first 14 coefficients b_n, c_n when $N = 2$. The values of b and c are approximately $b = 0.6928352170733$ and $c = 1.8185775134237$.

With necessary modifications a similar calculation was performed to obtain the singular solution f_∞, using the expansion (2.19) instead of (2.13). However, contrary to the regular solutions, there was no indication of the value of γ settling down to a limit when n was increased. A plot of these results is given in Figure 2.6.

It is known [Kuczma, 1968] that the only meromorphic function possessing a convergent Abel series is of the form (for some constant c)

$$h^*(u) = \frac{u}{1 - cu},$$

n	b_n	c_n
1	-0.3995352805231	0.2232406122039
2	0.6241085316034e-1	-0.1345161969489e-1
3	-0.1216873690647e-2	-0.7480773864376e-2
4	-0.4452135369632e-4	-0.5815968689412e-3
5	0.1692171258255e-5	0.5127800049896e-4
6	-0.6802373776624e-8	0.1429043993049e-4
7	-0.5999740035799e-9	0.8321253324135e-6
8	0.1268250709217e-10	-0.9878392016390e-7
9	0.9928052226669e-14	-0.1819800467726e-7
10	-0.4130427107048e-14	-0.3592277805731e-9
11	0.4642783291487e-16	0.1754423135696e-9
12	0.7173684857453e-18	0.1832070509061e-10
13	-0.2234733160494e-19	
14	0.5325272423571e-22	

Table 2.7: Coefficients of N = 2 regular solution of the FFE equation
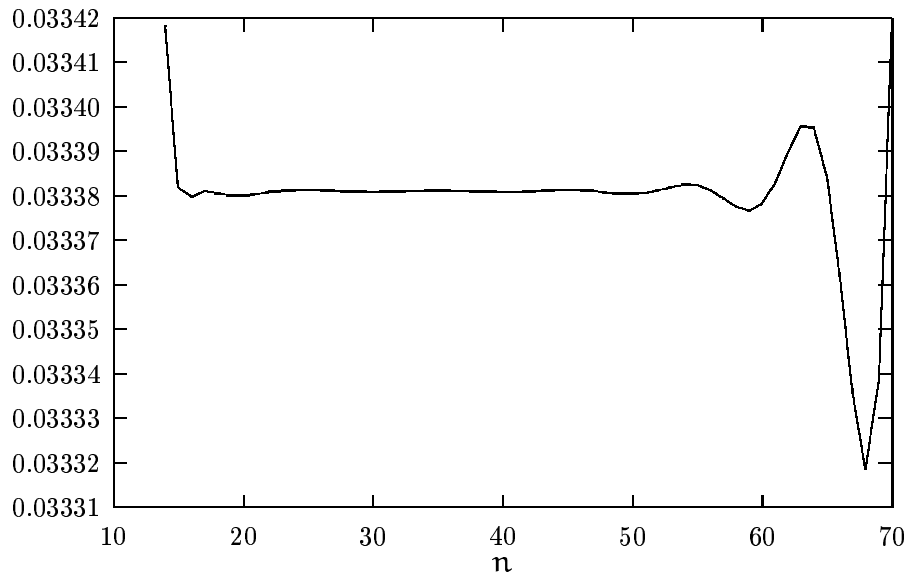
Figure 2.6: The approximation to the singular γ vs. series length n

with an associated Abel function $A(u) = -(cu)^{-1}$. As this h^* is not of this form, its Abel series A and the derived series f are divergent. Although this poses no obstacle when performing formal power series operations, we cannot expect meaningful results if we simply sum the divergent series f to determine the constants b, c and γ. The same problem arises if we wish to compose the divergent series with a power series containing a nonzero constant term. The divergent series, however, as shown by Eckmann and Wittwer [Eckmann and Wittwer, 1985], are Borel

summable. I now briefly explain how I have implemented Borel summation numerically, using an idea due to J.-J. Loeffel [Loeffel, 1976] and developed by [Sokal, 1980][1]. I make use of the Watson lemma, that under suitable growth conditions, if f possesses a divergent asymptotic expansion (as $z \to 0$), $f(z) \sim \sum_{k=0}^{\infty} f_k z^k$ then $B(t) = \sum_k f_k t^k / k!$ converges in some disk, and $f(z) = (1/z) \int_0^\infty \exp(-t/z) B(t) dt$. We may perform the Laplace integral by composing the f series with the conformal mapping $\phi(z) = -\beta \log(1 - z)$ for an appropriate real positive constant β and making use of the identities:

$$
\begin{aligned}
f(z) &= (1/z) \int_0^\infty \exp(-t/z) B(t) dt \\
&= (1/z) \int_0^\infty \exp(-t/z) \sum_k \hat{f}_k [\phi^{<-1>}(t)]^k dt \\
&= (1/z) \sum_k \hat{f}_k \int_0^\infty \exp(-t/z)(1 - \exp(-t/\beta))^k dt \\
&= \sum_k \hat{f}_k \frac{k!}{\prod_{j=1}^k (\beta/z + j)}
\end{aligned}
$$

where $\hat{f}_k$ are the Taylor coefficients of $f(-\beta \log(1 - z))$. I used this convergent representation of $f(z)$ for the shift of expansion point (that is, composition with an affine map), by simply substituting $z = b(1 - \gamma) - \gamma t$. See Appendix B for a listing of C++ code for this method. In Table 2.8 I give a table of values of the computed solution f.

I obtained in this way the behaviour shown in Figure 2.7. The numerical values

$$\gamma = 0.0333810598(\pm 5)$$

and $b = 0.391133000(\pm 2)$, were obtained with $\beta = 9$. (Here and later the quoted uncertainties refer to digits in the last place.) These considerably improve the best known numerical values [Eckmann and Wittwer, 1985]. It thus does seem very likely from this evidence that a singular solution with the stipulated analytic properties exists, which is consistent with the proof of Eckmann and Wittwer.

The limiting value of the unstable eigenvalue δ of (FFE) is also of interest. Several authors have estimated this singular δ from studies of iterated maps [Thompson and McGuire, 1988; van der Weele, Capel and Kluiving, 1987; Feingold, Gonzalez, Magnasco and Piro, 1991]. Eckmann and Epstein [Eckmann and Epstein, 1990]

---

[1] See also [Sokal, 1996] for a discussion of the relevance of these ideas to cultural studies.

x	f(x)	x	f(x)
0.00	1.000000	0.17	0.310200e-14
0.01	0.492580	0.18	0.176744e-16
0.02	0.225557	0.19	0.429463e-19
0.03	0.951872e-1	0.20	0.368697e-22
0.04	0.366539e-1	0.21	0.877693e-26
0.05	0.127320e-1	0.22	0.422290e-30
0.06	0.393676e-2	0.23	0.270164e-35
0.07	0.106692e-2	0.24	0.130805e-41
0.08	0.248909e-3	0.25	0.221327e-49
0.09	0.489402e-4	0.26	0.443674e-59
0.10	0.790972e-5	0.27	0.223401e-71
0.11	0.102015e-5	0.28	0.287125e-87
0.12	0.101355e-6	0.29	0.289672e-108
0.13	0.743558e-8	0.30	0.928416e-137
0.14	0.382695e-9	0.31	0.102114e-176
0.15	0.129853e-10	0.32	0.420735e-235
0.16	0.269163e-12		

Table 2.8: Values of the singular solution of the Feigenbaum equation

have given the best rigorous bounds: $29.5128 < \delta < 29.9571$. They also gave a numerical estimate $\delta = 29.5763$, but gave no description of how it was computed.

I have computed an estimate of the limiting δ using the algorithm of McGuire and Thompson [McGuire and Thompson, 1982]. We construct the approximants

$$\delta_n = X_{n+1}/X_n, \qquad n = 0, 1, 2, \ldots$$

where $X_0 = 1/f'(0)$, and

$$X_n = \gamma^{-n} \sum_{k=1}^{2^n-1} \frac{f^{<k>}(0)}{f^{<k>'}(0)}, \quad n > 0.$$

Here $f^{<k>}$ denotes the kth iterate of the solution to (FFE). This algorithm was implemented symbolically and simplified using (FFE) and its derivative. See [McGuire and Thompson, 1982] and [Thompson and McGuire, 1988] for a derivation of this formula. My best estimate (based on a 120 term series for f) is

$$\delta = 29.576303(\pm 1).$$
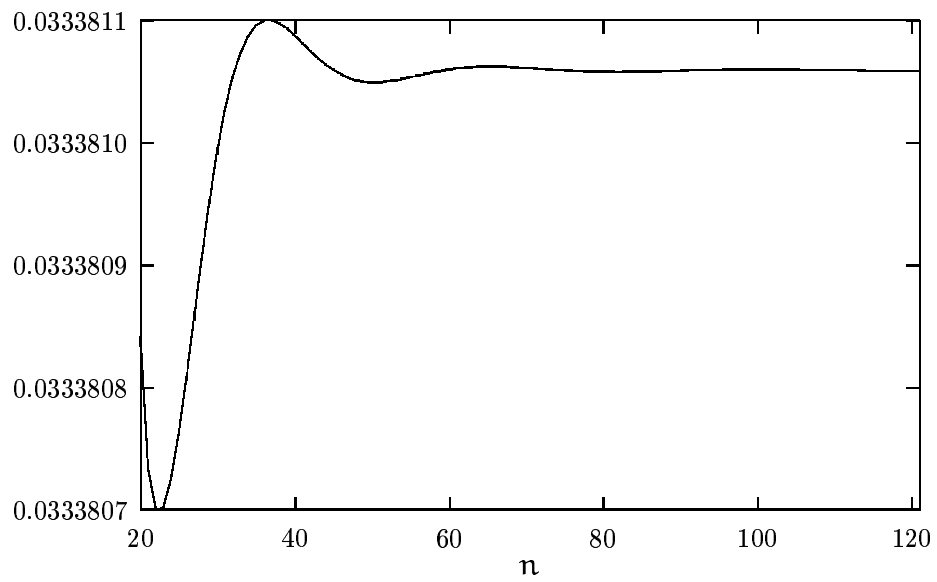
Figure 2.7: The approximation to the singular γ using Borel summation

Chapter 3

Feigenbaum scaling in complex maps

3.1 Generalized feigenvalues of mandelsets

In this section, I will discuss generalizations of Feigenbaum's constants α and δ to complex polynomial maps of degree higher than two, and present some numerical estimates. As in the real case, universality classes are found to depend on the nature of the critical points of the polynomial.

3.1.1 Introduction

Since the original discovery by Feigenbaum [Feigenbaum, 1979] of scaling properties of period-doubling in one-dimensional maps, there has been considerable interest in generalizing the concepts of Feigenbaum constants or feigenvalues δ and α which characterize the rate of parameter-dependent period doubling.

As described in § 2.1, the map

$$x \rightarrowtail f(x) \equiv \lambda - |x|^d, \qquad \lambda, x, d \in \mathbb{R}$$

has a cascade of bifurcations to 2^k-cycles at parameter values $\lambda_k, k = 1, 2, 3, \ldots$ which converge asymptotically geometrically to λ_∞ at a rate

$$\delta(d) = \lim_{k \to \infty} (\lambda_{k-1} - \lambda_k)(\lambda_k - \lambda_{k+1})^{-1}$$

37

with feigenvalue $\delta(d)$ depending on the order d of the critical point at zero.

The dependence of δ on d and corresponding universality classes of unimodal maps have been studied in some detail in this thesis and by [Derrida, Gervois and Pomeau, 1979; Collet et al., 1980; Chang, Wortis and Wright, 1981; McGuire and Thompson, 1982; Eckmann and Wittwer, 1985; Delbourgo and Kenny, 1986; van der Weele et al., 1986; van der Weele et al., 1987; Thompson and McGuire, 1988]. For unimodal maps, it is known that modulo power-law conjugacy [McGuire and Thompson, 1982], the feigenvalues depend solely on the exponent d of the critical point. The case of $\mathbb{R}^2$ maps has been discussed in Chapter 1.

Complex analytic maps of $\mathbb{C}$ and area-preserving maps of $\mathbb{R}^2$ are known to have extended universality classes and corresponding feigenvalues for general n-tupling, where $n = 2$ corresponds to period-doubling, etc. See also [Branner, 1988] for an extension to non-analytic complex maps. Cvitanović and Myrheim [Cvitanović and Myrheim, 1983], for example, studied n-tupling of the complex quadratic map

$$z \longmapsto \lambda - z^2, \qquad \lambda, z \in \mathbb{C} \tag{3.1}$$

and found feigenvalues $\delta_{m/n}$ for n-tupling with winding numbers m/n ($m = 1, 2$, ..., $n-1$) corresponding to eigenvalues crossing the unit circle at $\exp(2\pi i m/n)$. I denote this situation by (n/m)-tupling, omitting the m when it is clear from the context.

I wish to now describe scaling properties of higher degree polynomial mappings of the complex plane.

3.1.2   Complex analytic maps

I begin by considering the elementary polynomial maps of integer degree d.

$$z \longmapsto \lambda - z^d \equiv Q_{\lambda,d}(z), \qquad \lambda, z \in \mathbb{C} \tag{3.2}$$

which have a unique critical point (where the derivative of the map vanishes) at the origin. We are particularly interested in the complex parameter values λ_k corresponding to superstable n^k-cycles (that is, cycles containing the critical point) and the (n/m)-tupling feigenvalue

$$\delta_{m/n}(d) = \lim_{k \to \infty} (\lambda_{k-1} - \lambda_k)(\lambda_k - \lambda_{k+1})^{-1}. \tag{3.3}$$

We may now define the *mandelset* $M(f, c)$ to be the subset of the λ-plane for which the orbit (under an arbitrary complex polynomial f) of the critical point c of

f is bounded. Thus a polynomial has as many mandelsets as critical points. This generalizes the concept of the ordinary (quadratic) Mandelbrot set, which was first discovered by Brooks and Matelski [Brooks and Matelski, 1980]. This definition should be contrasted with that of the *connectivity locus* defined in [Branner, 1988] for cubic maps, where it is a subset of $\mathbb{C}^2$.

The λ sequences above thus lie in the mandelset. The limit points λ_∞ of such sequences lie on the boundary of the corresponding mandelset with intermediate parameter values for successive n-tupling located at the points of contact of asymptotically self-similar components of the Mandelbrot set. The self-similarity has been discussed by Milnor [Milnor, 1989]. Diagrams of some of these generalized mandelsets are shown in [Briggs, Quispel and Thompson, 1991]. It is trivial to show that the mandelset $M(\lambda - z^d, 0)$ has $(d-1)$-fold rotational symmetry.

Some numerical feigenvalues $\delta_{m/n}(d)$ and $\alpha_{m/n}(d)$ for (n/m)-tupling of equation (3.2) are given in Table 2.5 with $\delta_{m/n}(d)$ defined by [Cvitanović and Myrheim, 1983] and, for a map f_λ with critical point c of order d,

$$\alpha_{m/n}(d) = \lim_{k \to \infty} \left[c - f_{\lambda_k}^{<n^k>}(c) \right] \left[c - f_{\lambda_{k+1}}^{<n^{k+1}>}(c) \right]^{-1} \tag{3.4}$$

where $f_\lambda^{<N>}$ denotes f_λ composed with itself N times and λ_k is the parameter value corresponding to the superstable n^k-cycle.

More complicated polynomial maps with more than one critical point can also be studied by the above methods. For example, the map

$$z' = \lambda + z^4(20z^2 - 48z + 30)$$

has two critical points: at zero with $d = 4$ and at one with $d = 3$. The computed feigenvalues $\delta_{m/n}(d)$, $\alpha_{m/n}(4)$ are given in Table 3.1. In this table c is the critical point, d its degree and n the tupling value. For $n = 2$, $m = 1$; and for $n = 3$, $m = 1$ or 2. Notice the agreement with the elementary $d = 3$ and $d = 4$ values.

3.1.3 Feigenbaum equations for n-tupling

The renormalization group method [Feigenbaum, 1979] can also be applied to complex maps [Cvitanović and Myrheim, 1983] with universality classes of functions corresponding to n-tupling obtained from solutions of the functional equation

$$g(z) = \alpha g^{<n>}(z/\alpha) \tag{3.5}$$

map	d	c	n	α	δ
$\lambda - z^2$	2	0	2	-2.502	4.6692
		0	3	$-2.0969 - 2.35828i$	$4.600 + 8.981i$
$\lambda - z^3$	3	0	2	$-0.2518 + 1.8647i$	$3.031 - 4.4556i$
		0	3	$-1.3812 - 2.0149i$	$15.795 + 0.5758i$
$\lambda - z^4$	4	0	2	-1.6903	7.2847
		0	3	$-1.9768 - 0.6356i$	$13.035 + 17.795i$
$\lambda - z^5$	5	0	2	$-1.157 + 1.099i$	$7.851 - 5.347i$
$\lambda + z^4 \times$	3	1	2	$-0.252 + 1.86i$	$3.031 - 4.456i$
$(20z^2 - 48z + 30)$			3	$-1.3812 - 2.0147i$	$15.795 + 0.5758i$
	4	0	2	-1.69	7.2847
			3	$-1.97 - 0.635i$	$13.035 + 17.795i$

Table 3.1: Feigenvalues computed directly from equations (3.3) and (3.4)

where $g^{<n>}$ denotes g composed with itself n times and $g(0) = 1$. For (n/m)-tupling

$$\alpha = \alpha_{m/n} = \left[g^{<n-1>}(1) \right]^{-1}$$

where $\alpha_{m/n}$ is defined by equation (3.4) and the corresponding $\delta_{m/n}$ can be computed from g by an asymptotic functional iteration process or from a related Feigenbaum functional equation [van der Weele et al., 1987; Cvitanović and Myrheim, 1983].

Solutions of equation (3.5) satisfying

$$g(z) = 1 + g_1 z^d + \mathcal{O}(z^{2d}) \tag{3.6}$$

yield feigenvalues for universality classes of functions having $z = 0$ as a critical point with exponent d. Direct substitution of (3.6) into (3.5) provides one method of solution. An alternative successive approximation method, due to [van der Weele et al., 1987], begins with a zeroth order approximation of the form $g(z) = 1 + g_1 z^d$. This method exploits the representation of the solution g as

$$g(z) = \lim_{k \to \infty} g_k(z)$$
$$g_k(z) \equiv \mu \alpha^k f_{\lambda_\infty}^{<n^k>}(z/(\mu \alpha^k)),$$

where $f_\lambda(z) \equiv 1 - \lambda z^d$. This, together with the relations (for $n = 2$)

$$g(0) = 1$$
$$g(1) = \alpha^{-1} \tag{3.7}$$
$$g'(1) = \alpha^{d-1}$$

suffices to determine μ, α, and λ_∞ and hence $g(z)$ for period-doubling. I have solved equations (3.7) by using a Newton-Raphson iteration in three-dimensional complex space to find μ, α, and λ_∞ in the case $n = 2$. It was found necessary to use multiple-precision arithmetic, principally because a large number of iterations of f (n^k, k typically up to ten) are involved, in which round-off error must be kept under control. With intermediate calculations to about fifty decimal places, typically four or five correct decimal places are obtained in the final α. Note the agreement between Tables 3.1 and 3.2. There are no real solutions for odd d. In Table 3.2, d is the order of the critical point of the map f, and the last column examines a relation suggested by [Delbourgo and Kenny, 1986].

d	α	δ	$\frac{(2d-1)\delta}{(2d-2)\alpha^d}$
2	-2.502907875	4.669	1.118
3	$-0.25181 \pm 1.864695i$	$3.303 \mp 4.45i$	$1.01 \pm 0.25i$
4	-1.690302	7.28	1.06
	0.60556 ± 1.50946	$2 \mp 5.7i$	$0.97 \pm 0.14i$
5	$0.8978 \pm 1.1812i$	$1.3 \mp 6.3i$	$0.51 \pm 1.5i$
	$-1.1568 \pm 1.09914i$	$7.85 \mp 5.34i$	$1.03 \pm 0.06i$
6	-1.4677	9.3	1.09
	$1.015 \pm 0.9536i$	$0.9 \mp 6.4i$	$0.0042 \pm 0.056i$
	$-0.5188 \pm 1.4061i$	$6.3 \mp 8.4i$	$1.07 \pm 0.06i$
7	$1.067 \pm 0.794i$	$0.7 \mp 6.5i$	$0.74 \pm 0.35i$
	$-1.12537 \pm 0.7232i$	$11.1 \mp 6.0i$	$1.17 \pm 0.06i$
	$-0.0699 \pm 1.4338i$	$5.0 \mp 10.7i$	$1.013 \pm 0.1i$
8	-1.35798	11	1.02
	$1.091 \pm 0.677i$	$4.7 \mp 6.6i$	$0.68 \pm 0.89i$
	$0.23619 \pm 1.3633i$	$4.0 \mp 12i$	$1.02 \pm 0.08i$
	$-0.8950 \pm 1.0766i$	$10.086 \mp 8.98i$	$1.02 \pm 0.04i$
9	$1.102 \pm 0.5897i$	$0.4 \mp 6.6i$	$0.90 \pm 0.32i$
	$0.444 \pm 1.2668i$	$3.0 \mp 1.3i$	$1.06 \pm 0.04i$
	$-0.5445 \pm 1.2416i$	$8.7 \mp 1.19i$	$1.04 \pm 0.03i$
	$-1.250 \pm 0.5314i$	$13.5 \mp 6.77i$	$1.04 \pm 0.01i$

Table 3.2: Feigenvalues computed from the functional equation with $n = 2$

It is more difficult to compute δ. I used two methods; that of van der Weele et al. [van der Weele et al., 1987], and that of McGuire and Thompson [McGuire and Thompson, 1982]. The latter was usually more rapidly convergent. Table 3.2 gives a synthesis of the two methods, based on my experience of the numerical behaviour of these methods. The results, although inaccurate, are sufficient to confirm agreement with Table 3.1.

One interesting feature of this method is that for fixed $d \geqslant 4$ and $n = 2$ one obtains multiple complex solutions. Some typical results are given in Table 3.2. In this particular case it appears that the number of distinct solutions of equation (3.5) (with $n = 2$) of the form (3.6) is the integral part of half d. I have confirmed for the case $d = 4, 5$ and 6, that the points λ_∞ appear graphically to lie on the boundary of the mandelset. These solutions are further studied in the next section.

Asymptotic forms of feigenvalues for large n [Cvitanović and Myrheim, 1983] and large d have been suggested by [Delbourgo and Kenny, 1986]. In particular, the asymptotic relation

$$\lim_{n \to \infty} \frac{(2d - 1)\delta_{m/n}}{(2d - 2)\alpha_{m/n}^d} = 1$$

suggested in [Delbourgo and Kenny, 1986] for real maps appear to be quite accurate in the complex case even for $n = 2$ (see the last column of Table 3.2, and also Table 3.3).

3.1.4   Conjectures

Table 3.2 suggests the following conjectures
1. $\text{sign}(\text{Im } \alpha) = -\text{sign}(\text{Im } \delta)$
2. $\lim_{d \to \infty} \frac{\delta}{\alpha^d} = 1$
3. $\lim_{d \to \infty} |\alpha(d)| = 1$
4. $\Re\delta > 0$.

for which I know of no theoretical justification, and which perhaps deserve further investigation.

3.2   Properties of complex universal functions

I will here examine the properties of complex analytic solutions to Feigenbaum's functional equation for period n-tupling, and the associated Cantor sets.

3.2.1   Introduction

The scaling properties of period-doubling of real maps, first noticed by Feigenbaum [Feigenbaum, 1979], are now well understood. In most respects, the results have been made rigorous [Lanford, 1982; Vul et al., 1984; Epstein, 1986; Jiang, Morita and Sullivan, 1992; Pollicott, 1991]. The same cannot be said about n-tupling

phenomena ($\mathbb{Z} \ni n = 2, 3, 4, \ldots$) which occur in parameterized families of complex polynomial maps.

I will study the following situation. Let f_a be a polynomial depending analytically on the complex parameter a. Recall that in § 3.1.2, I defined for each critical point c of f_a, the *mandelset* $M(f_a, c)$ to be the set of a values such that the orbit of c is bounded under iteration of f_a. By an n-tupling sequence ($n = 2, 3, 4, \ldots$) I mean a sequence of parameter values $\{a_k, k = 0, 1, 2, \ldots\}$, such that f_{a_k} possesses a superstable n^k-cycle. Each such a_k is a centre of a *hyperbolic component* of $M(f_a, c)$, that is, a connected region for which f_a possesses a stable n-cycle of constant period. It is observed that for each sequence $\{a_k\}$ of centres of adjacent hyperbolic components of the mandelset of f, the sequence $\{(a_{k-1} - a_k)/(a_k - a_{k+1})\}$ is convergent to a value δ. The same is true for certain sequences which do not correspond to adjacent hyperbolic components, such as those lying along filaments in the mandelset. To distinguish different period n-tupling sequences for the same n, one may use the winding number of the n-cycle with respect to the fixed point [Cvitanović and Myrheim, 1989], or some symbolic labelling scheme [Bai-lin, 1989].

By *universality*, I mean the conjecture that the constants δ depend only on the degree d of the critical point c.

Only the case of quadratic complex polynomials has been studied in detail [Cvitanović and Myrheim, 1989]. Ideally, one would like a general proof of universality, along the lines of that of that of [Epstein, 1986] for the real case. However, a reasonable conjecture (inspired by the analogy with the real case), is that all universal properties are encoded in the complex analytic solutions of the Feigenbaum functional equation:

$$g(z) = \alpha g^{<n>}(\alpha^{-1}z), \qquad g(0) = 1, \qquad \alpha^{-1} \equiv g^{<n>}(0), \qquad (3.8)$$

where $g^{<n>}$ indicates the n-fold composition of g with itself. Thus Cvitanović and Myrheim [Cvitanović and Myrheim, 1989] studied the solutions of this equation which behave like

$$g(z) = 1 + g_1 z^2 + \mathcal{O}(z^4)$$

for small $|z|$. In this section I will compute other solutions and examine their properties. In particular, solutions of the form

$$g(z) = 1 + \sum_{k=1}^{\infty} g_k z^{kd}, \qquad (2 \leqslant d \in \mathbb{Z})$$

describe universal features of period n-tupling for polynomials with a critical point of degree d.

3.2.2   Numerical Methods

It has been proven that (3.8) possesses real analytic solutions for even d [Epstein, 1986; Lanford, 1982]. To my knowledge, there is no proof of existence of complex solutions. However, numerical evidence is strong that there are solutions for each integer $d \geqslant 2$. Previously, complex numerical solutions to (3.8) have been computed for $d = 2$ by a Newton-Raphson iteration [Press et al., 1986] in a certain space of complex polynomials. These polynomials then approximate the first terms of the Taylor expansion of g. I have found a similar method successful for $d > 2$ also. To implement this method, it is first necessary to compute the linearization of the period n-tupling operator ($n = 2, 3, 4, \dots$) [1]

$$
\begin{aligned}
(T^{(n)}f)(z) &\equiv \alpha_f f^{<n>}(\tilde{z}) \\
\alpha_f^{-1} &\equiv f^{<n>}(0) \\
\tilde{z} &\equiv f^{<n>}(0)z.
\end{aligned}
$$

I will use the notation $\tilde{z} \equiv \alpha^{-1}z$ throughout, suppressing the dependence of α on n where it is clear from the context. It is convenient also to define

$$
(\tilde{T}^{(n)}f)(z) \equiv f^{<n>}(z),
$$

that is, T without the α scaling.

I will denote the composition of functions by simple juxtaposition. I indicate the linearization of T about an arbitrary function f by L_f. To preserve the normalization $g(0) = 1$, L_g must act on functions h satisfying $h(0) = 0$. For example, for the case of period-doubling ($n = 2$), we have

$$
(L_f h)(z) = \alpha_f h(1)[zf'(\tilde{z})f'f(\tilde{z}) - \alpha_f^2 ff(\tilde{z})] + \alpha_f f'f(\tilde{z})h(\tilde{z}) + \alpha_f hf(\tilde{z}).
$$

To compute the general case ($n > 2$), I first need a few lemmas. Denoting the linearization of $T^{\widetilde{(k)}}$ about f by $\tilde{L}_f^{(k)}$, we have

Lemma 1

$$
\tilde{L}_f^{(n)}h(z) = hf^{<n-1>}(z) + f'f^{<n-1>}(z)\tilde{L}_f^{(n-1)}h(z).
$$

---

[1]This could have been done with algorithmic differentiation methods (Appendix B), but for reasons of efficiency I chose to make the exact calculation above.

Proof by induction on n ∎.

Lemma 2 *Suppose that* $f = g + \epsilon h$ *is near a solution* g *of (3.8). Then*

$$(\alpha_f^{(n)})^{-1} = g^{<n>}(0) + \epsilon \left[\sum_{i=1}^{n-2} hg^{<i>}(1) + h(1)(g^{<n-1>})'(1)\right]/g'(1) + \mathcal{O}(\epsilon^2).$$

Proof by induction on n. For $n = 2$ we have $\alpha = g(1) + \epsilon h(1)$. Now

$$
\begin{aligned}
(\alpha_f^{(n+1)})^{-1} &= (g + \epsilon h)^{<n+1>}(0) \\
&= (g + \epsilon h)\alpha_f^{(n)^{-1}} \\
&= (g + \epsilon h)\left[g^{<n>}(0) + \epsilon(\sum_{i=1}^{n-2} hg^{<i>}(1) + h(1)(g^{<n-1>})'(1)/g'(1)\right] \\
&= g^{<n+1>}(0) + \epsilon \left[\sum_{i=1}^{n-1} hg^{<i>}(1) + h(1)(g^{<n>})'(1)\right]/g'(1) + \mathcal{O}(\epsilon^2) \ \blacksquare
\end{aligned}
$$

Next I will to compute the linearization of T. We have

$$
\begin{aligned}
T^{(n)}(f + \epsilon h)(z) &= \alpha_{f+\epsilon h}^{(n)} \tilde{T}^{(n)}(f + \epsilon h)(\alpha_{f+\epsilon h}^{(n)^{-1}} z) \\
&= \alpha_{f+\epsilon h}^{(n)}\left[f^n(\alpha_{f+\epsilon h}^{(n)^{-1}} z) + \epsilon\tilde{L}^{(n)}(f + \epsilon h)(\alpha_{f+\epsilon h}^{(n)^{-1}} z)\right] \\
&\quad + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

which gives

$$(L_f^{(n)} h)(z) = \left[\alpha_f^{(n)} z(f^{<n>})'(\tilde{z}) - (\alpha_f^{(n)})^2(\tilde{z})\right] \times$$

$$\left[\sum_{i=1}^{n-2} hf^{<i>}(1) + h(1)(f^{<n-1>})'(1)/f'(1)\right] \alpha_f^{<n>}(\tilde{L}_f^{(n)} h)(\tilde{z}).$$

This result for the linearization of the period n-tupling operator allows us to compute iteratively $L_f^{(n)}$ for any n. Approximate solutions to the Feigenbaum functional equation can now be readily computed by the following procedure. I first compute a 2-term approximation $g(z) = 1 + g_1 z$ by inserting it in the Feigenbaum functional equation and solving the appropriate polynomial equation. For example, for period-tripling ($n = 3$), we have

$$d^2(1 + g_1(1 + g_1)^d)^{d-1}(1 + g_1)^{d-1}g_1^2 - 1 = 0.$$

Each solution g_1 of this equation gives rise to an initial approximation for a Newton iteration in the space of coefficients $\{g_1, g_2, \ldots, g_m\}$, where typically m is from 20 to 40. To refine this approximation, I must compute the action of T and L on

basis elements z^k ($k = 1, 2, \ldots, m$), and thus represent these operators by $m \times m$ matrices. The Newton-Raphson iteration

$$g \leftarrowtail g - (Lg - I)^{-1}(Tg - g)$$

can then be performed by standard methods of LU decomposition [Golub and Van Loan, 1989]. I generally used the technique of increasing the number of coefficients in the expansion of g one by one, iterating until convergence for each case. Otherwise some solutions can be missed, because the initial two-term approximation is not sufficiently close to a solution. I have listed all the solutions found for $d = 2, 3, \ldots, 8$ in Tables 3.5 to 3.11. Of course, α is immediately obtainable from g, but δ is a little more difficult to determine accurately. I used both the power method to obtain the largest eigenvalues of the matrix representing L, as well as a packaged algorithm [Smith, Boyle, Klema and Moler, 1970] to compute all eigenvalues. In all cases, both methods agreed. Some of the corresponding feigenvalues provide a more accurate determination than those computed from orbits in § 3.1; others are new.

3.2.3   Asymptotics

The numerical difficulties of this type of computation increase rapidly with d and n. I was not able to exceed $d = 8$. However, this was enough to discern the interesting trends visible in Figures 3.1 and 3.2. It appears that there is a 'principal' sequence for which α^d tends to a limit about $-2 - 7.16i$, and a secondary sequence with a limit less easily estimated.

I also checked the conjecture of [Delbourgo, 1992] that suggested that

$$\delta/\alpha^d \sim \frac{d-1}{d-1/2}.$$

This was indeed accurately obeyed, as shown in Table 3.3, though I am aware of no rigorous justification for this asymptotic relation.

3.2.4   Theory

In this section I will outline a theory for complex universal functions based the case of real quadratics as presented in [Vul et al., 1984]. For $d = n = 2$, equation (3.8) has a locally unique solution [Lanford, 1982]. It possesses the following properties, which have been established by numerical estimates with rigorous bounds ([Koch et al., 1994]).

d	$(2d-1)\delta/((2d-2)\alpha^d)$
2	1.1180
	1.2625 + 0.8463i
	0.9628
	0.9198
3	1.0120 − 0.2426i
	0.9493 + 0.1330i
	0.9924 + 0.0139i
4	0.9721 + 0.2905i
	1.0411
	0.9304 + 0.1611i
	1.0213 + 0.1112i
	0.9912 + 0.0168i
	1.0138
	0.9625 + 0.0559i
5	0.9487 + 0.3096i
	1.0311 + 0.0616i
6	0.9334 + 0.3187i
	1.0198 + 0.0867i
	1.0228
7	0.9226 + 0.3235i
	1.0160 + 0.1019i
	1.0242 + 0.0248i
8	0.9145 + 0.3262i
	1.0215 + 0.0439i
	1.0113 + 0.1119i

Table 3.3: Test of Delbourgo's conjecture

1. g is an even, unimodal, convex function

2. $|g| < 1$ for $|z| < 1$.

3. g has a single unstable fixed point in $|z| < 1$.

We now construct the sets

$$\Delta_0^k \equiv [\alpha^{-k}, -\alpha^{-k}], \qquad k = 1, 2, 3, \ldots$$
$$\Delta_i^k \equiv g^{<i>}(\Delta_0^k), \qquad i = 1, 2, \ldots, 2^k - 1.$$

Then the set

$$\mathcal{F} \equiv \overline{\bigcap_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} \Delta_k^n}$$
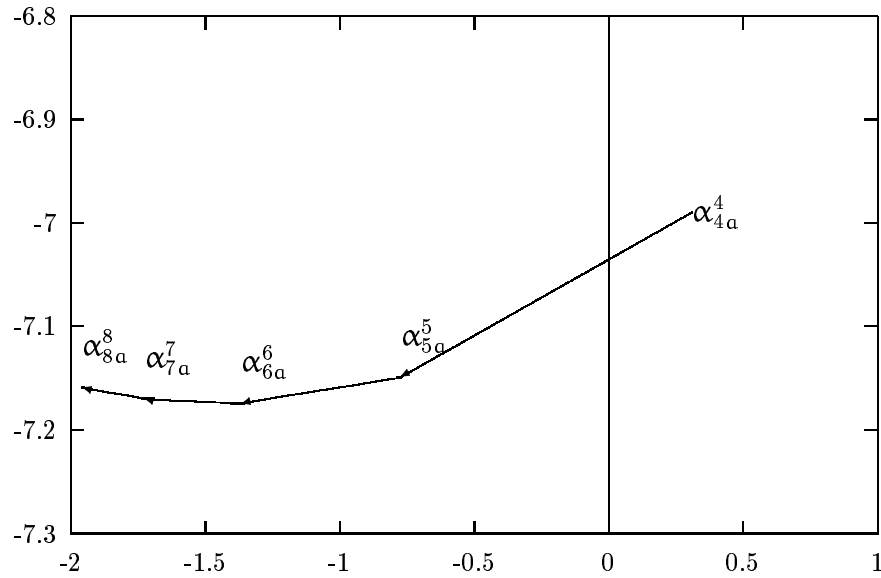
Figure 3.1: A sequence of powers α^d for period-doubling
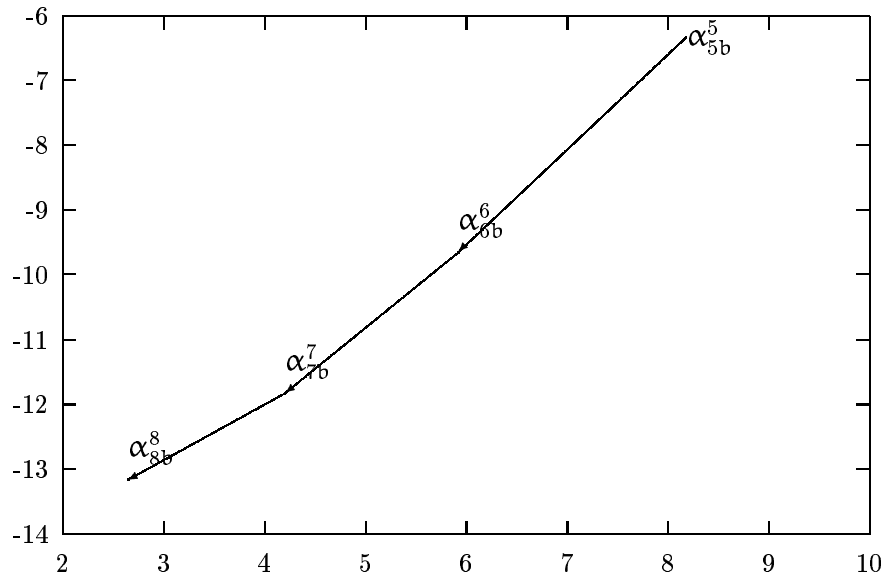
Figure 3.2: Another sequence of powers α^d for period-doubling

is shown in [Vul et al., 1984] to be a Cantor set, which is invariant and attractive under g. Additionally, g restricted to $\mathcal{F}$ possesses an invariant measure with respect to which g is ergodic, and a Hausdorff dimension $D_H \approx 0.538$. I emphasize that these results can be considered rigorous, but that they do depend on numerical estimates.

Unfortunately, in the general complex case these properties cannot all be satisfied. For example, we can no longer have $|g| < 1$ in any neighbourhood of 0, since $g(0) = 1$ and g is analytic. Also, it appears that the intervals Δ_k^n cannot be

replaced by simple sets such as disks. Nevertheless, if I define $A = \{g^{<k>}(0), k = 0, 1, \ldots, n-1\}$ and $A_m = \{z \in A, z = g^{<k>}(0), k \equiv m(\mathrm{mod}\, n)\}$, then we have a disjoint partition of A, with $A_k = g(A_{k-1}), (k = 1, 2, \ldots, n-1)$. (This forms a Markov partition.) We may now define surjective analytic maps

$$f_k = \alpha g^{<(n-k)\mathrm{mod}\, n>} : A_k \to A$$

for $k = 0, 1, \ldots, n-1$.

The sets A_k allow us to assign a symbol sequence $I_1 I_2 \cdots$ to each orbit of g, where $I_j = k$ iff the jth element of the orbit belongs to A_k. By a *prime word*, I mean one that cannot be written as a product of shorter cycles of length > 1. Figure 3.3 depicts the situation for period-tripling ($n = 3$).
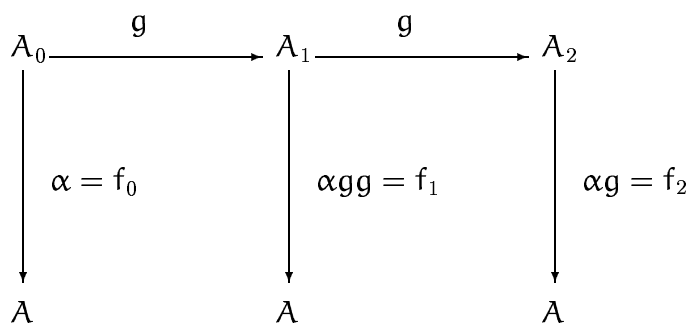
Figure 3.3: The dynamics of g for period-tripling

All the numerical solutions g computed by the method above possess the following properties:

1. The orbit of the origin is bounded.

2. For each prime word, there is an orbit of g with the corresponding symbolic dynamics.

3. The maps f_k are expanding on A_k.

I have found the last condition to be easily satisfied by all the solutions g that I have computed to the Feigenbaum functional equation.

With these properties satisfied, I will apply the ζ–function formalism [Artuso, Aurell and Cvitanović, 1990a; Artuso, Aurell and Cvitanović, 1990b] to compute topological entropies, Hausdorff dimensions, etc.

3.2.5 ζ–function methods

In this section I use the technique of symbolic dynamics to generalize the ζ–function method of [Artuso et al., 1990a], to complex maps. The (Ruelle) ζ–function is defined as

$$\zeta^{-1}(z) = \prod_p (1 - z^{n_p} t_p).$$

Here the product is over prime cycles, n_p is the number of prime cycles of length $|p|$, and t_p depends on the application to be made of the ζ–function [Artuso et al., 1990a]. For example, I will calculate the generalized dimension spectrum $D(q)$ by putting $t_p = 1/|\Lambda_p|^q$, where Λ_p is the stability of the prime cycles with word p.

For each solution g of the Feigenbaum functional equation that I have found for period-doubling and period-tripling, I computed all the prime cycles to length 6 and 4 respectively. This was achieved by a Newton iteration, for which the initial approximation was found by linearizing the appropriate composition of functions f_k as defined above, and solving the resultant linear system. That is, defining the 'centre' c_k of A_k by $c_k = g^{<k>}(0)$, we have that the linearization $\tilde{f}_k(z) = a_k z + b_k$ of f_k at c_k is uniquely determined by

$$
\begin{aligned}
\tilde{f}_k(c_k) &= 1 \\
\tilde{f}_k'(c_k) &= f_k'(c_k) = \alpha \prod_{i=0}^{k-1} g' g^{<i>}(g^{<k>}(0)),
\end{aligned}
$$

so that $f_k(z - c_k) = \tilde{f}_k(z - c_k)) + \mathcal{O}((z - c_k)^2)$. Then the zero of the ζ–function was found by a standard bisection algorithm [Press et al., 1986]. The resulting functions $D(q)$ were normalized by dividing by the known exact value $D(-\infty) = \alpha$ to facilitate comparison. The results for $d = 2, 3, 4$ are shown in Figure 3.4.

3.2.6   Conjectures

In [Cao and Peng, 1992], Cao and Peng have claimed to have shown a further universality phenomenon; namely that the normalized $D(q)$ curves are independent of the integer n specifying the n-tupling. However, I find a small difference (visible in Figure 3.4) which I believe is not attributable to numerical imprecision or insufficient length of my ζ–function expansions. However, the differences are less at larger d, and the Hausdorff dimensions appear to converge to a universal value, approximately 0.7, independent of n, for large d.
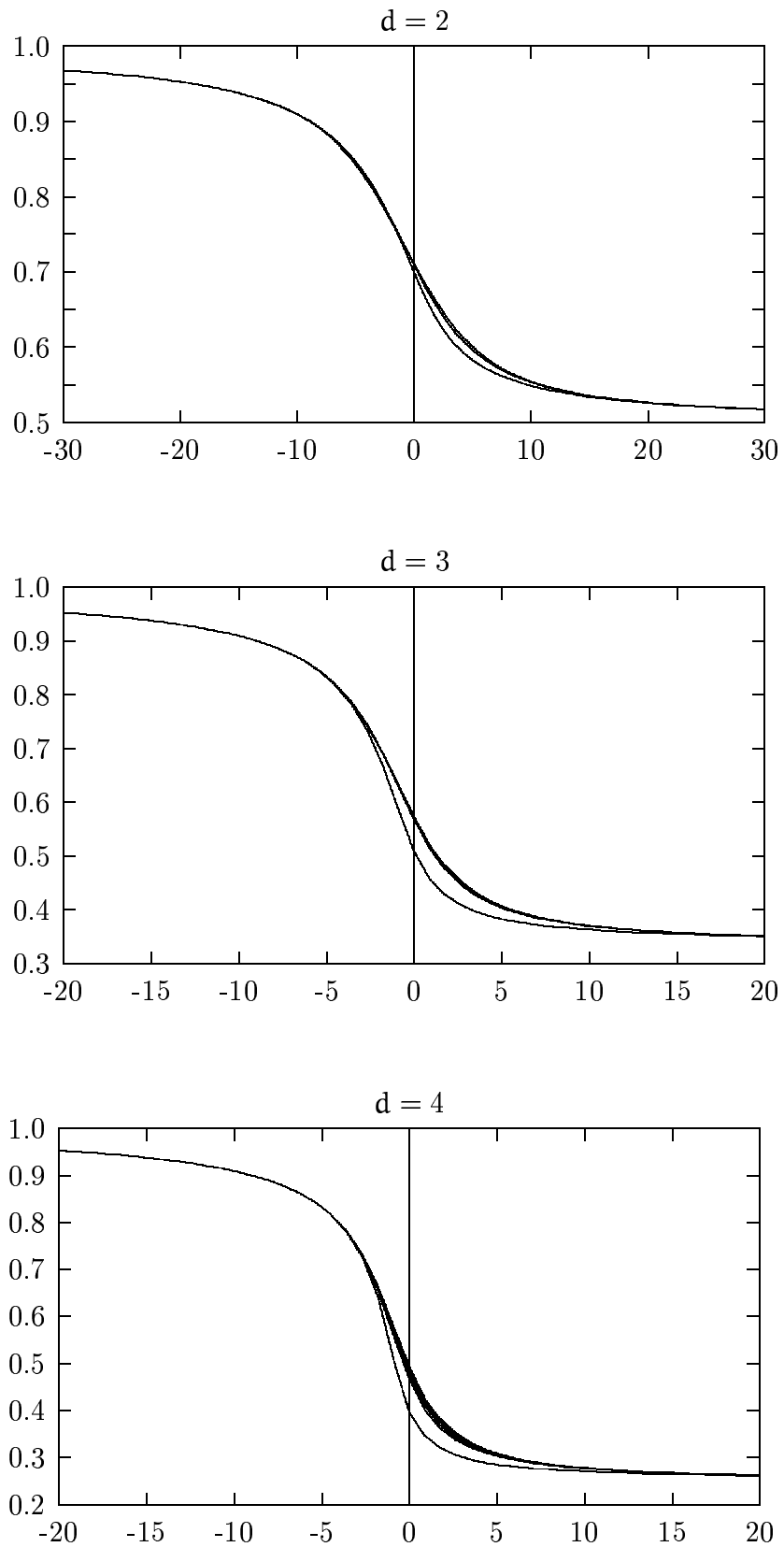
Figure 3.4: Normalized dimension spectra vs. q for d = 2, 3, 4

#	n = 2	n = 3
1	0	0
2	1	1
3	10	2
4	100	10
5	110	20
6	1000	21
7	1100	100
8	1110	200
9	10000	110
10	11000	210
11	10100	120
12	11100	220
13	10110	112
14	11110	221
15	100000	1000
16	110000	2000
17	101000	1100
18	111000	2100
19	101100	1200
20	111100	2200
21	100110	2010
22	101110	1110
23	111110	2110
24		1210
25		2210
26		2011
27		2120
28		1220
29		2220
30		2111
31		2211
32		2212

Table 3.4: Prime words to length 6 for period-doubling, and to length 4 for period-tripling

n	$d = 2$			
2	α	=	-2.5029078758	
	δ	=	4.6692016098	
	D_H	=	0.53805	
3	α	=	-2.0969198891	$+2.3582796414i$
	δ	=	4.6002246075	$-8.9812258410i$
	D_H	=	0.6705	
3	α	=	-9.2773411158	
	δ	=	55.2470265888	
	D_H	=	0.3503	
4	α	=	-38.8190742971	
	δ	=	924.0840897220	

Table 3.5: Computed α and D_H for $d = 2$

n	$d = 3$			
2	α	=	-0.2518110538	$-1.8646953210i$
	δ	=	3.3030825654	$+4.4555653650i$
	D_H	=	0.6304	
2	α	=	-1.3812446902	$+2.0149778332i$
	δ	=	15.7949041805	$-0.5758145585i$
	D_H	=	0.701	
3	α	=	0.5550478650	$-2.0774298839i$
	δ	=	-1.1739000990	$+11.769570584i$
	D_H	=	0.7327	
3	α	=	1.0637879021	$+4.18128728881i$
	δ	=	-35.1902921171	$-50.5411484731i$
	D_H	=	0.428	
3	α	=	-1.0063122570	$-3.9742979387i$
	δ	=	36.4878477641	$+40.7713104377i$
	D_H	=	0.445	

Table 3.6: Computed α and D_H for $d = 3$

n		$d = 4$		
2	α	=	0.6055533369	+1.50946943801
	δ	=	2.0014174483	-5.74661617881
	D_H	=	0.669	
2	α	=	-1.6903029710	
	δ	=	7.2846862172	
	D_H	=	0.643	
3	α	=	-1.9768080555	+0.63567083361
	δ	=	13.0355603861	-17.7951408151
	D_H	=	0.75	
3	α	=	1.0500981936	+1.42543356121
	δ	=	-3.1339521925	-11.8989991571
	D_H	=	0.77	
3	α	=	2.0473566198	+2.23048871641
	δ	=	-64.0574900213	-22.8450337821
	D_H	=	0.47	
3	α	=	-2.8862481542	+0.81697470571
	δ	=	38.8250937062	-59.7948341891
	D_H	=	0.49	
3	α	=	0.7724764592	-2.73443235831
	δ	=	24.2205911707	+49.8135850291
	D_H	=	0.506	
3	α	=	-3.1521573432	
	δ	=	85.7916290911	
	D_H	=	0.471	
3	α	=	1.7157307246	+2.82490466701
	δ	=	-51.9232855393	+83.8406761191
	D_H	=	0.45	

Table 3.7: Computed α and D_H for $d = 4$

3.3   The area of the mandelset

I conclude this chapter with an amusing speculation on the area A of the (ordinary quadratic) mandelset $M(z^2 + c, 0)$ defined in § 3.1. Around 1993 there was considerable interest in this question, stimulated by a distributed internet computation (based on pixel counting) organized by Yuval Fisher and Jay Hill [Fisher and Hill, 1993]. Previous to this, there had been some rigorous bounds computed in [Ewing and Schober, 1992], which, however, were done by extremely slowly convergent series methods. [Ewing and Schober, 1992] obtained the area bounds $1.3744 < A < 1.7274$, whereas [Fisher and Hill, 1993] obtained $1.5031197 < A < 1.5613027$.

It is important to emphasize that this is not a natural question. The area of the Mandelbrot set is not a holomorphic invariant; indeed the area of domains in complex analysis cannot be expressed without making use of the complex conjugate, a non-analytic concept.

Nevertheless, I could not resist making an independent contribution to the debate in 1993, and I proceeded by computing very accurately the area of all regions of period less than 9. I achieved this by calculating with Maple the polynomials defining the boundaries of the hyperbolic components, and then doing a numerical area integration (by triangles) giving results accurate to at least 8 decimal places. I obtained the results of Table 3.12.

I was impressed by the fact that these data, when plotted on log scales, fell very accurately on a straight line. I still have no idea why such a behaviour should occur. The slope estimated by least-squares was -2.8002. That this trend continues to higher periods can be seen on the graph in Figure 3.5 of the results of [Fisher and Hill, 1993].
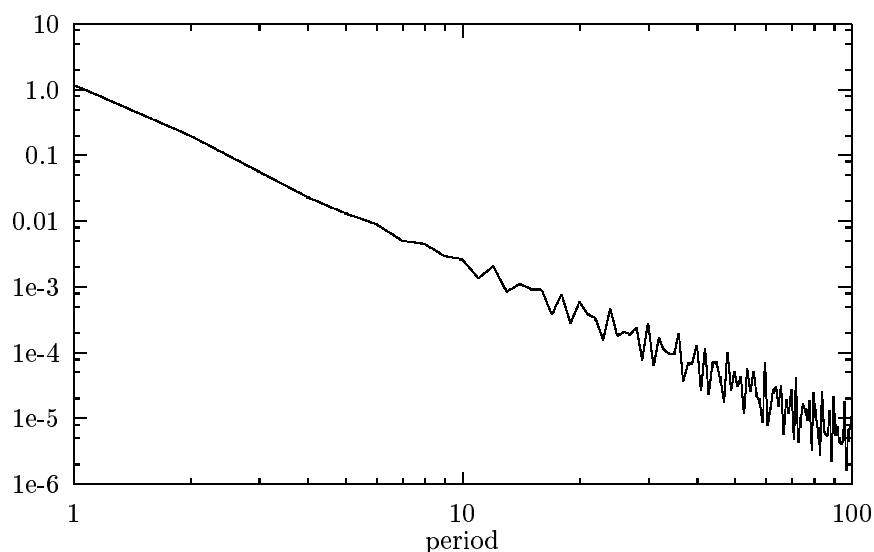
Figure 3.5: The area of mandelset components vs. period

Assuming that the area of all periods greater than 8 may be estimated by a tail correction based on the assumption that this trend is correct, I obtained a total area estimate of 1.49969. I suspect this is more accurate than Fisher and Hill's pixel-counting, which must always over-estimate the area. It suggests the intriguing conjecture that the exact area is 3/2. However, this question is not important enough to spend any more time on.

n			$d = 5$	
2	α	=	0.8978292495	+1.1812649293i
	δ	=	1.3142736789	−6.2421902164i
	D_H	=	0.67	
2	α	=	−1.1568197586	+1.0991412850i
	δ	=	7.8514513431	−5.3472539555i
	D_H	=	0.648	

Table 3.8: Computed α and D_H for $d = 5$

n			$d = 6$	
2	α	=	1.0153549578	+0.9535499148i
	δ	=	0.9185291354	−6.4838695018i
	D_H	=	0.66	
2	α	=	−0.5187961798	+1.4061105086i
	δ	=	6.2543342897	−8.4914536674i
	D_H	=	0.69	
2	α	=	−1.4677425684	
	δ	=	9.2962417948	
	D_H	=	0.7	

Table 3.9: Computed α and D_H for $d = 6$

n			$d = 7$	
2	α	=	1.0674967558	+0.7938858694i
	δ	=	0.6709769811	−6.6211335484i
	D_H	=	0.7	
2	α	=	−0.0699141311	+1.4338098843i
	δ	=	5.0540369278	−10.706982710i
	D_H	=	0.7	
2	α	=	−1.2537057995	+0.7232678107i
	δ	=	11.0619587435	−6.0001888906i
	D_H	=	0.66	

Table 3.10: Computed α and D_H for $d = 7$

n	d = 8			
2	α	=	1.0914792239	+0.6776782577i
	δ	=	0.5059767933	−6.7073667712i
	D_H	=	0.69	
2	α	=	−0.8950592016	+1.0765825059i
	δ	=	10.8510657319	−8.9839510639i
	D_H	=	0.70	
2	α	=	0.2361759477	+1.3633424681i
	δ	=	3.8765989539	−12.1543320288i
	D_H	=	0.7	
2	α	=	−1.3581941166	
	δ	=	10.9524291007	
	D_H	=	0.7	

Table 3.11: Computed α and D_H for d = 8

period	area/π	cumulative total
1	.3749999995	.3749999995
2	.0624999999	.4374999994
3	.0179980106	.4554980100
4	.0074087064	.4629067164
5	.0041747693	.4670814857
6	.0028440321	.4699255178
7	.0016099799	.4715354977
8	.0014222118	.4729577095

Table 3.12: The area of mandelset components by numerical integration

Chapter 4

Scaling in circle maps

Jaques: *''Tis a Greek invocation, to call fools into a circle'*

As You Like It, act 2, scene 5

In the first part of this chapter I discuss corrections to Feigenbaum-Kadanoff-Shenker scaling in circle maps. I then discuss the question of the asymptotic limits of the Feigenbaum-Kadanoff-Shenker α and δ constants. Finally, I discuss a generalization of Manton-Nauenberg scaling in the boundary of Siegel domains.

4.1 Introduction

In this chapter I will study the Feigenbaum-Kadanoff-Shenker scaling in the critical family of circle maps on $\mathbb{R}$:

$$\theta \mapsto f_\Omega(\theta) = \Omega + \theta - \frac{1}{2\pi}\sin(2\pi\theta). \tag{4.1}$$

This case (degree of inflection point equal to three) has been much studied ([Tzeng, Yu, Hu and Hu, 1991; Kadanoff, 1981; Shenker, 1982; Feigenbaum, Kadanoff and Shenker, 1982; Jensen, Bak and Bohr, 1983; Cvitanović, Shraiman and Soderberg, 1985; Cvitanović, Gunaratne and Vinson, 1990], but here I wish to generalize to different degrees of the point of inflection, and to discuss corrections to scaling. The average advance in the angle θ per iteration is called the winding number w. Denoting the nth iterate of f as $f^{<n>}$, we can define w_f as the limit:

$$w_f(\Omega) = \lim_{n\to\infty} \frac{f_\Omega^{<n>}(\theta_0) - \theta_0}{n} \tag{4.2}$$

for some initial θ_0. If $w_f(\Omega) = p/q$ is rational, where p and q are positive integers, the map represents mode-locked behaviour with period q.

From the work of Denjoy [Denjoy, 1932], Arnol'd [Arnol'd, 1963; Arnol'd, 1965], Herman [Herman, 1979] and Yoccoz [Yoccoz, 1984], we know that if f is invertible, then $w(\Omega)$ is a monotonic increasing function of Ω and is independent of the initial value θ_0. It is, however, constant on a dense set of subintervals, and its graph forms a so-called 'devil's staircase'.

It is well known that every irrational number w can be represented by an infinite regular continued fraction

$$w = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}}, \qquad (4.3)$$

and that the sequence of best rational approximations p_n/q_n to an irrational number w is found by truncating the continued fraction:

$$p_n/q_n = [a_1, a_2, \ldots, a_n]. \qquad (4.4)$$

This is the classical Euclidean algorithm.

Considering now the case of the sine map defined in equation (4.1), we may define a sequence $\{\Omega_n\}$ by means of

$$f_{\Omega_n}^{<q_n>}(0) = p_n,$$

so that there is a q_n-cycle containing zero with winding number p_n/q_n. This scheme was first proposed by Greene [Greene, 1968; Greene, 1979].

I first consider the golden mean (gm) winding number, which has a continued fraction expansion

$$w_{gm} = \cfrac{1}{1 + \cfrac{1}{1 + \cdots + \cfrac{1}{1 + \cdots}}} = [1, 1, 1, \ldots].$$

By successively truncating this infinite fraction, one generates a sequence of rational approximations $1/1, 1/2, 2/3, 3/5, \ldots$ to w_{gm}. The nth term in this sequence is f_{n-1}/f_n where the f_n are the Fibonacci numbers. These are defined recursively by

$$f_{n+1} = f_n + f_{n-1}; \qquad n = 1, 2, 3, \ldots, \qquad f_0 = 0, \qquad f_1 = 1.$$

From this it follows that

$$\lim_{n \to \infty} \frac{f_{n-1}}{f_n} = w_{gm}. \tag{4.5}$$

From the corresponding sequence $\{\Omega_n\}$, we may define the Feigenbaum(-Kadanoff-Shenker) δ by

$$\delta \equiv \lim_{j \to \infty} \frac{\Delta \Omega_j}{\Delta \Omega_{j+1}}$$

which has the approximate value -2.834. There is also an orbit scaling: if ϕ_k is the value of the nearest cycle element to zero in the q_k cycle, then

$$\alpha \equiv \lim_{k \to \infty} \frac{\phi_k}{\phi_{k+1}}$$

exists, and is about -1.289. I now will compute more accurate scaling laws, following the method of § 2.3.

4.2 Corrections to scaling

I postulate the following forms for the behaviour of $\Delta \Omega_k$ and ϕ_k:

$$\Delta \Omega_k = \sum_{i=0}^{\infty} \frac{a_i}{\delta_i^k}$$

$$\phi_k = \sum_{i=0}^{\infty} \frac{b_i}{\alpha_i^k}.$$

To determine the validity of these expansions, I used a polynomial map behaving like $\text{sign}(\theta)|\theta|^d$ near zero for each value of d. I accurately computed from four to eight of the quantities δ_i and α_i for about 50 values of d between 1 and 35. Here δ_i and α_i are the constants to be determined, with $\delta_0 = \delta$ and $\alpha_0 = \alpha$. The exponents are named in order of increasing magnitude. It is necessary to compute typically 25 parameter values Ω_k, for which I used a Newton-Raphson iteration method. I then computed the exponents by the method described in § 2.3.2, always using high precision arithmetic. The results are shown in Figures 4.1 and 4.2. In these figures, the solid line is the dominant scaling exponent. The results are not very different from those previously obtained for maps of the real line in § 2.3, but are much more difficult to compute accurately.
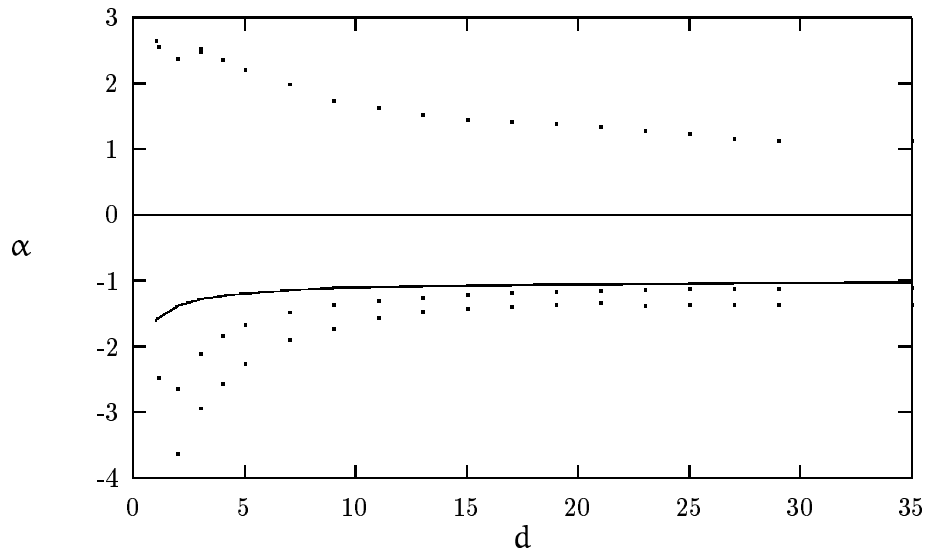
Figure 4.1: The scaling exponents α_i vs. d at the golden mean
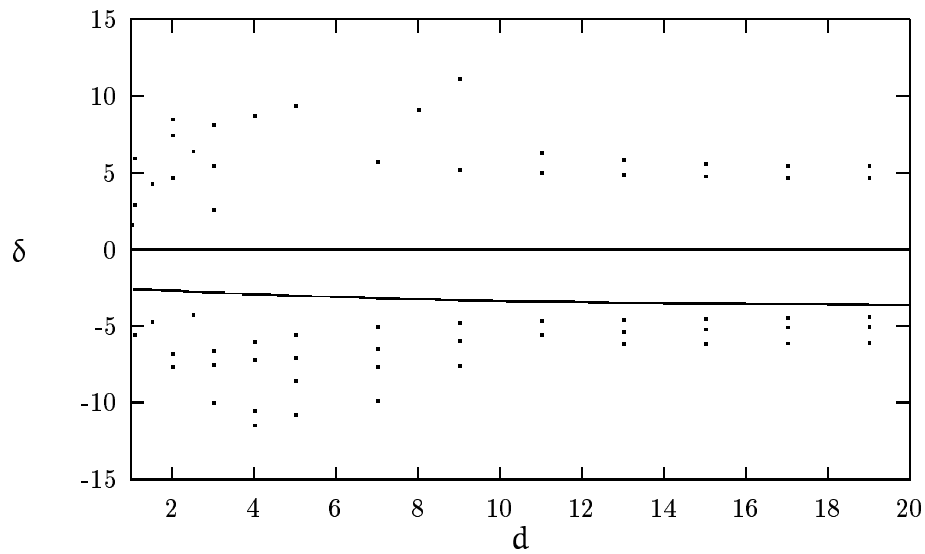
Figure 4.2: The scaling exponents δ_i vs. d at the golden mean

4.3   The Feigenbaum-Kadanoff-Shenker functional equation

While the Feigenbaum functional equation characterizes the period-doubling route
to chaos, the Feigenbaum-Kadanoff-Shenker equation [Feigenbaum et al., 1982]

$$g \circ g(\epsilon^2 x) = \epsilon g(x), \quad \epsilon < 0 \qquad \text{(FKS)}$$

characterizes the quasiperiodic route to chaos. This transition is modelled by a
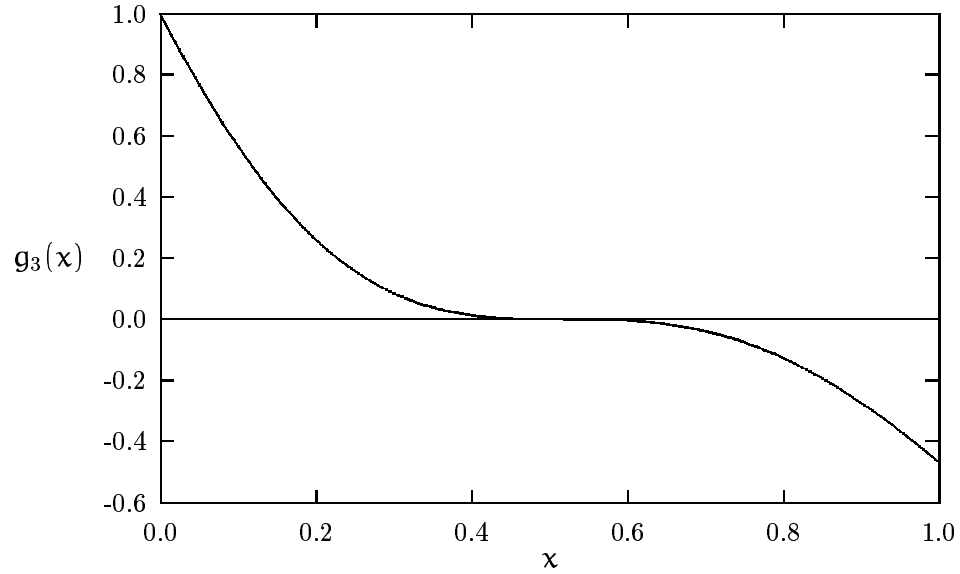
Figure 4.3: The solution of FKS for N = 3

system consisting of two coupled oscillators with irrational frequency ratio, in this case, the golden mean $(\sqrt{5}-1)/2$. For background to this subject, see [Feigenbaum et al., 1982; Ostlund, Rand, Sethna and Siggia, 1983; Rand, Ostlund, Sethna and Siggia, 1982]. Note that ϵ is here the reciprocal of the constant α used by [Feigenbaum et al., 1982]. Though a large amount of research has been devoted to this situation, nothing has been done on the asymptotics of the functional equation, which I propose to discuss in this section.

Using the standard normalization, the graph of g decreases monotonically from $g(0) = 1$ to its inflection point $g(b) = 0$ for some $0 < b < 1$ and then decreases monotonically to $g(1) = \epsilon$. The existence of solutions has been shown by Mestel [Mestel, 1985], Lanford [Lanford, 1984] for $N = 3$, and Eckmann and Epstein [Eckmann and Epstein, 1986] for $N = \infty$.

Performing the same power conjugacy transformation as for (FFE), one can proceed to define and denote the principal solutions of order N by g_N and the singular solution by g_∞. See Figure 4.3 for a plot of the function g_3.

We thus have

$$g(0) = 1, \quad g(b) = 0, \quad g(1) = \epsilon$$

where $0 < \epsilon^2 < |\epsilon| \leqslant b < 1$ for $N \geqslant 1$, and

$$g \circ g(\epsilon^2) = \epsilon^2, \quad g(\epsilon^2 b) = b, \tag{4.6}$$

from the substitution of $x = 1$ and $x = b$ respectively into (FKS).

Since the solution may have a singularity at b, we must consider two distinct analytic functions on either side of b. Let $v(x)$ be the restriction of g to $b < x \leqslant 1$ (g will denote the restriction to $0 \leqslant x < b$) then (FKS) takes the form

$$g \circ g(\epsilon^2 x) = \epsilon v(x), \quad \text{for} \ \ b < x \leqslant 1 \tag{4.7}$$

and

$$v \circ g(\epsilon^2 x) = \epsilon g(x), \quad \text{for} \ \ 0 \leqslant x < b. \tag{4.8}$$

We can eliminate v from (4.7) and (4.8) by setting $x = g(\epsilon^2 y)$, $0 \leqslant y < b$ (hence $b < x \leqslant 1$); then (FKS) becomes

$$g \circ g(\epsilon^2 g(\epsilon^2 y)) = \epsilon^2, \qquad g(y) \ \ 0 \leqslant y < b, \tag{FKS1}$$

an equation which only involves $g : [0, b) \to (0, 1]$.

Differentiating (4.7) and (4.8) and substituting $x = 1$ and $x = 0$ respectively we find (assuming $g'(0) \neq 0$ and $g'(0) \neq \infty$) that

$$\epsilon^2 g' \circ g(\epsilon^2) g'(\epsilon^2) = 1. \tag{4.9}$$

Any solution of (FKS1) which satisfies $g(0) = 1$ and hence (4.6) must satisfy (4.9).

I now perform a construction exactly analogous to that in § 2.4.2, to which reference may be made for more detail. I denote by g_k the restriction of g to the interval $I_k = [\epsilon^{2k+2} b, \epsilon^{2k} b)$, $k = 0, 1, 2, \ldots$, then $g(\epsilon^2 y) = g_{k+1}(\epsilon^2 y) > b$ for $y \in I_k$ (since $g(\epsilon^2 b) = b$), $\epsilon^2 g(\epsilon^2 y) > \epsilon^2 b$, $g(\epsilon^2 g(\epsilon^2 y)) = g_0(\epsilon^2 g_{k+1}(\epsilon^2 y))$ and $g \circ g_0(\epsilon^2 g_{k+1}(\epsilon^2 y)) = \epsilon^2 g(y) < \epsilon^2 = g \circ g(\epsilon^2)$. Hence $g_0(\epsilon^2 g_{k+1}(\epsilon^2 y)) > \epsilon^2 b$ and (FKS1) becomes

$$g_0 \circ g_0(\epsilon^2 g_{k+1}(\epsilon^2 y)) = \epsilon^2 g_k(y) \ \ \text{for} \ \ y \in I_k \tag{FKS2}$$

and in particular

$$g_0 \circ g_0(\epsilon^2 g_1(\epsilon^2 y)) = \epsilon^2 g_0(y) \ \ \text{for} \ \ y \in I_0 = [\epsilon^2 b, b). \tag{FKS3}$$

As with (FFE) the problem is to determine g_0 so that g_1 is be the analytic continuation of g_0. Then, (FKS2) is an equation for the analytic continuation of g_0 over the whole interval $[0, b)$. I now focus solely on (FKS3).

Since $g(\epsilon^2 b) = b$ and g is supposed to be analytic at $\epsilon^2 b$, we may set

$$g(\epsilon^2 b + \epsilon^2 t) = b + b_1 t + b_2 t^2 + \cdots \quad \text{with} \ \ b_1 < 0$$

and assume convergence of the series for some $|t| < r$, $r > 0$. Next set $\psi(x) = g(\epsilon^2 x)$, $h(x) = \psi(\psi(x))$; then, writing g for both g_0 and g_1) (FKS3) takes the form

$$g \circ h(x) = \epsilon^2 g(x), \tag{S}$$

that is, Schröder's functional equation for h. As $h(b) = b$, we again transform the fixed point to $u = 0$ by defining $h^*(u) = b - h(b - u)$, $g^*(u) = g(b - u)$; then $h^*(0) = 0$ and

$$g^* \circ h^*(u) = \epsilon^2 g^*(u). \tag{S*}$$

We have arrived at exactly the same functional equation as when we were solving (FFE). The procedure for solving (S*) for g^* is almost exactly as outlined earlier but with a few differences. The Taylor expansion of $h^*(u)$ takes the same form as (2.12). If b_1^2 is not one then (S*) has a convergent solution of the form

$$g^*(u) = cu^N(1 + c_1 u + c_2 u^2 + \cdots) \tag{4.10}$$

and the coefficient of u^N gives $b_1^{2N} = \epsilon^2$,

$$\epsilon = -|b_1|^N, \quad b_1 = -|\epsilon|^{1/N}.$$

The constant c in (4.10) is determined from the condition $g^*(b - \epsilon^2 b) = b$, while b is adjusted so that $g^*(b) = 1$. Substituting $u = b(1 - \epsilon^2) - \epsilon^2 t$ into (4.10) we can transform back to centre $\epsilon^2 b$ and obtain

$$g(\epsilon^2 b + \epsilon^2 t) = b + b_1^* t + b_2^* t^2 + \cdots \tag{4.11}$$

and equate the coefficients b_k^* with the original coefficients b_k. See table 4.1 for a range of regular ϵ and b values obtained by solving the Schröder equation derived from (FKS).

If b_1^2 is one, it is again necessary to use Abel's functional equation instead of Schröder's. The procedure for solving Abel's equation in this instance is exactly as that set out for solving the singular (FFE) equation with the exception that the substitution $u = b(1 - \epsilon^2) - \epsilon^2 t$ is used to transform back to centre $\epsilon^2 b$ and the conditions $g(\epsilon^2 b) = b$ and $g \circ g(\epsilon^2) = \epsilon^2$ serve to determine c and ϵ.

The singular series solution for the (FKS) equation shows the same divergent properties that characterize the singular (FFE) solution. However, after applying

N	n	ϵ	b	$-g'_N(0)$
1	5	−0.6180339887	0.6180339887	0.6180339887
2	23	−0.5188837225	0.5496018421	0.9052865510
3	29	−0.4673813162	0.5178627069	1.0728988866
4	39	−0.4349284204	0.4991137721	1.1844018525
5	49	−0.4123238695	0.4866077115	1.2645217061
6	59	−0.3955620152	0.4776221430	1.3251266285
7	71	−0.3825838027	0.4708314423	1.3726956939
8	89	−0.3722105156	0.4655075010	1.4110920617
9	98	−0.36371399	0.46121511	1.44277302

Table 4.1: Properties of regular solutions of the FKS equation
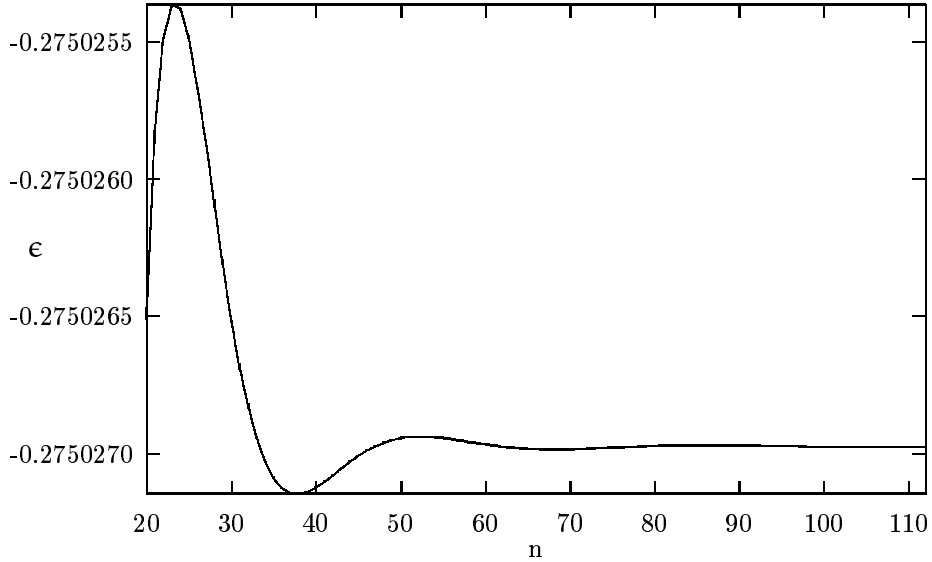
Figure 4.4: The approximation to the singular FKS ϵ vs. series length n

the Borel summation techniques as used in § 2.4.3, good numerical stability is achieved, and I can confidently state that the singular FKS ϵ is

$$-0.275026971(\pm 3),$$

and the value of b is $0.420056641(\pm\ 3)$. Figure 4.4 illustrates the behaviour of the singular FKS ϵ as more coefficients are added to the expansion of g.

Regarding the question of the Feigenbaum δ for this situation, Dixon [Dixon, 1997] has derived an analogous procedure to the McGuire-Thompson method of § 2.4.3. Dixon obtained an estimate $\delta \approx -4.121326(\pm 4)$. Now Hu et al. [Hu, Valinia and Piro, 1990] have previously obtained a value of -4.11 from orbit studies of the

non-smooth map

$$f_\Omega(t) = \Omega + \frac{1}{2}\left[t + \text{sign}(t)\exp\left(\frac{3}{2} - \frac{3}{8t^2}\right)\right].$$ (4.12)

As I suspected that the discontinuous derivatives of this map at the endpoints of the interval were affecting the results, I constructed the following C^∞ map the interval $[-\frac{1}{2},\frac{1}{2})$:

$$f_\Omega(t) = \Omega + \text{sign}(t)\frac{e}{4}\left(1 + \frac{1}{4t^2}\right)\exp\left(\frac{-1}{4t^2}\right)$$ (4.13)

which has an essential singularity at the origin. From orbit calculations (taken to the 29th Fibonacci number) of this map, I estimated $\delta \approx -4.121$ using Padé analysis techniques, in better agreement with Dixon's value than the results of [Hu et al., 1990]. See also [Dixon, Kenny and Briggs, 1997] for more recent results.

4.4 Manton-Nauenberg scaling

Manton-Nauenberg scaling occurs on the boundary of Siegel domains, that is, on the boundary of regions on which maps of the form

$$f(z) = \exp(2\pi i\Omega)z + \mathcal{O}(z^2)$$ (4.14)

are conjugate to their linear part. Inside the Siegel domain, orbits have quasiperiodic motion on smooth Siegel curves which are analogous to the KAM tori in area-preserving maps. However, at the boundary, the curves become non-smooth (indeed, fractal) and the scaling is different, as described below. The connection with circle maps in § 4.1 will be apparent.

Siegel considered linearization of maps of the form of equation (4.14) by means of Schröder's functional equation (see Chapter 7), and when Ω is diophantine, so that we have

$$f(\sigma(z)) = \sigma(\exp(2\pi i\omega)z)$$

and

$$|\Omega - p/q| > \lambda/q^\mu$$

for all integers $m, n \geqslant 1$ and for some positive $\lambda, \mu \in \mathbb{R}$. Here σ is the Schröder function of f.

The Manton-Nauenberg scaling then arises when we consider a sequence p_n/q_n of rational approximants converging to Ω. If these are generated with the Euclidean algorithm (4.4), then we observe that

$$f^{<q_n>}(z_c) \sim z_c + A\alpha^{-n}$$

as $n \to \infty$. Here z_c is a critical point of f (that is, $f'(z_c) = 0$), and A and α are constants, depending on Ω only . Manton and Nauenberg [Manton and Nauenberg, 1983] found that for Ω equal to the golden mean ϕ, $|\alpha|$ was about 1.34783. In the golden mean case, we have $q_n = f_n$, the nth Fibonacci number. This was confirmed by [Widom, 1983] and [Osbaldestin, 1992], and later Stirnemann [Stirnemann, 1994b; Stirnemann, 1994a] proved rigorously that this scaling law is indeed correct.

My purpose in this section is twofold: firstly, to show that the estimation of α by ratios is incorrect in [Osbaldestin, 1992] in this situation; and secondly to investigate the corrections to scaling problem, with an analysis analogous to that in § 2.3. A new phenomenon will be found to occur: the presence of two complex scaling exponents of equal modulus. This does not occur in any n-tupling situation, and its occurrence in this situation has apparently been missed by previous investigators.

Rather than using the polynomial map f, we may more generally consider the rational map of degree $d \in \mathbb{Z}^+$

$$f(z) = \frac{cz^d + b}{az^d + 1}.$$

Osbaldestin [Osbaldestin, 1992] then defines α by $\lim_n z_{q_{2n}}/z_{q_{2n-1}}$ and shows that

$$\lim_n z_{q_{2n-1}}/z_{q_{2n}} = \overline{\lim_n z_{q_{2n}}/z_{q_{2n+1}}}.$$

I have computed orbits (for $c = exp(2\pi i\phi)$, $d = 2$ and $a = 0$) and made a Padé analysis to recover the scaling exponents, as described in § 2.3.2. It is quite clear that there are *two* dominant scaling exponents $\alpha = \pm 1.347831995$. Note that these are real, and equal absolute value. How then, do we explain Figure 2 in [Osbaldestin, 1992] which shows a nonzero imaginary part to α ?

It is because in this case, the ratio method does *not* recover α correctly, only $|\alpha|$. We can see this as follows. If

$$z_{q_n} \sim A\alpha^n + B(-\alpha)^n + \cdots \qquad \text{as } n \to \infty$$

then

$$\frac{z_{q_n}}{z_{q_{n-1}}} \sim \frac{A\alpha^n + B(-\alpha)^n}{A\alpha^{n-1} + B(-\alpha)^{n-1}} + \cdots$$

$$= \alpha \frac{A + (-1)^n B}{A - (-1)^n B} + \cdots$$

$$= \begin{cases} \alpha \frac{A+B}{A-B} + \cdots & n \text{ even} \\ \alpha \frac{A-B}{A+B} + \cdots & n \text{ odd} \end{cases}.$$

Thus if $|(A + B)/(A - B)| = 1$, the ratio method will compute the correct $|\alpha|$, but the computed phase will involve the uninteresting amplitudes A and B, which is not as desired.

The 'phase' of α observed in the ratio method is, however, correctly predicted from the Padé amplitude. We may compute the Padé amplitude by L'Hôpital's rule: Let $p(x)/q(x)$ be a Padé approximant to $\sum z_{q_n} x^n$; then for each root r of $q(x)$ we compute:

$$\lim_{x \to r} \frac{d[(x - r)p(x)]/dx}{rq'(x)}.$$

I found numerically (with a [6,6] Padé approximant) that for $r = 1.347831995$, this gives an amplitude $A = -0.400145347 + 1.287064i$, and $r = -1.347831995$, the amplitude is $B = -0.400145347 - 1.287064i$ (which is Osbaldestin's complex α) with $(A + B)/(A - B) = 0.9999997$.

I next used Padé approximants to study variation of α spectrum with d, and $a = 0$. It unambiguously shows that the dominant scaling is always

$$z_{q_n} \sim A\alpha^n + B(-\alpha)^n$$

with A, B complex, α real, and $|(A - B)/(A + B)| = 1$. If this is the case, the ratio method will give

$$\lim z_{q_n}/z_{q_{n-1}} = \alpha(A \pm B)/(A \mp B)$$

which explains the observed odd/even behaviour, but shows that the nonzero phase of α is purely an artifact of the inappropriate use of the ratio method to estimate α. In other words, the ratio method *assumes* that the scaling exponent of smallest modulus in unique. If it is not, the result has amplitude information mixed in.

Arneodo and Holschneider [Arneodo and Holschneider, 1988] have predicted on the basis of the thermodynamic formalism that $|\alpha(d)|^d = |\alpha(1/d)|$. This is confirmed by my approximate data, as shown in Table 4.2.

d	$|\alpha|$	$|\alpha|^d$
1/8	1.902372	
8		1.895929
1/4	1.884893	
4		1.884626
1/2	1.816651	
2		1.816623

Table 4.2: The Arneodo and Holschneider conjecture
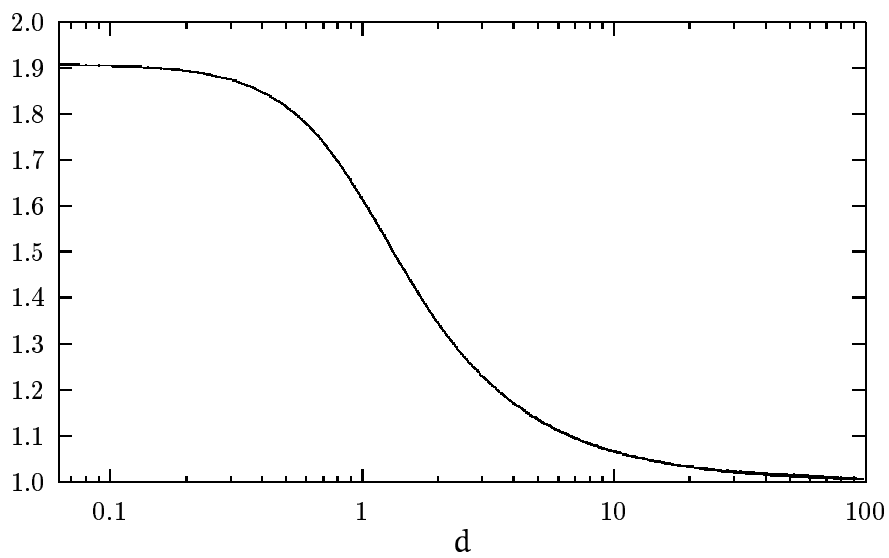
Figure 4.5: The absolute value of the scaling exponent α_0 vs. d
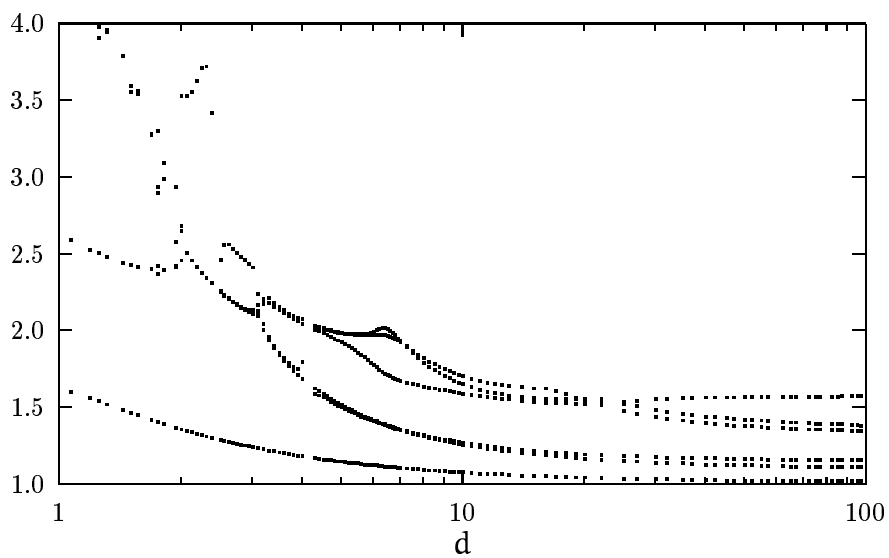
Figure 4.6: The absolute value of the scaling exponents α_i vs. $d, i = 0, \ldots, 12$

Figure 4.7: The real part of the scaling exponents α_i vs. $d, i = 0, \ldots, 12$
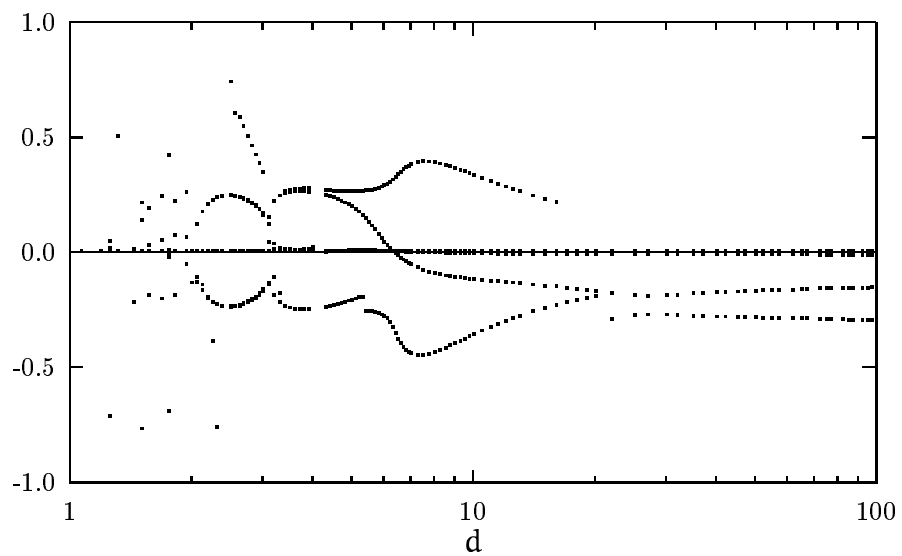
Figure 4.8: The imaginary part of the scaling exponents α_i vs. $d, i = 0, \ldots, 12$

Chapter 5

Scaling in torus maps

<div align="right">

Othello: *'Chaos is come again'*
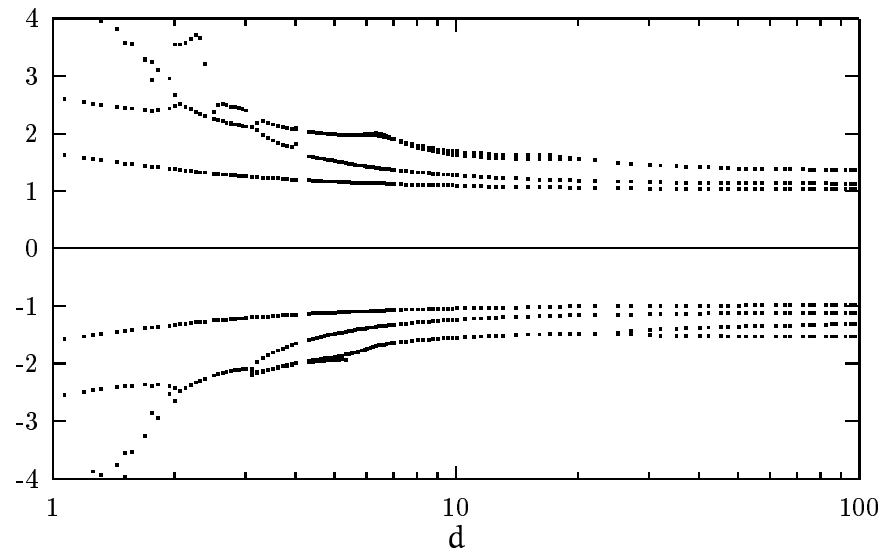
Othello, act 3, scene 3

</div>

In this chapter I will discuss the generalization of the ideas on scaling in circle maps (Chapter 4) to the two-torus $\mathbb{T}^2$. This will require a discussion of simultaneous rational approximation methods and two-dimensional analog of the Siegel linearization theorem. In the investigation I found several potentially useful lines of enquiry not mentioned in the existing literature. I have made use of these results in two ways: firstly, to generalize the Feigenbaum-Kadanoff-Shenker circle map scaling, and secondly, to try to generalize the Manton-Nauenberg scaling (§ 4.4).

5.1   Maps of the two-torus

I wish to examine the dynamics of families of maps

$$
\begin{bmatrix} x \\ y \end{bmatrix} \rightarrowtail f_\Omega \left(\begin{bmatrix} x \\ y \end{bmatrix} \right)
$$

of the plane $\mathbb{R}^2$, parameterized by $\Omega \in \mathbb{R}^2$, which satisfy

$$
f_\Omega \left(\begin{bmatrix} x + m \\ y + n \end{bmatrix} \right) = f_\Omega \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) + \begin{bmatrix} m \\ n \end{bmatrix},
$$

for all integers m, n. By identifying opposite edges of the square $(-1/2, 1/2) \times (-1/2, 1/2)$, we can consider this as a map of the two-torus $\mathbb{T}^2$. This will model

three-frequency mode-locked systems, as described in [Hu and Mao, 1987]. If for some x, y, Ω and positive integers p_1, p_2, q we have

$$f_\Omega^{<q>}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + p_1 \\ y + p_2 \end{bmatrix},$$

then we say the orbit has *rotation vector*, (or, loosely, winding number) $(p_1, p_2)/q$. The bifurcation behaviour of such maps has been fully described by [Baesens, Guckenheimer, Kim and MacKay, 1991].

In [Hu and Mao, 1987], Hu and Mao attempted to generalize the Feigenbaum-Kadanoff-Shenker circle map scaling by considering a map of the form

$$f_\Omega\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + \Omega_x - \frac{K}{2\pi}\sin(2\pi x)\cos(2\pi y) \\ y + \Omega_y + \frac{K}{2\pi}\sin(2\pi x) \end{bmatrix} \tag{5.1}$$

and a polynomial approximation to it:

$$f_\Omega\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + \Omega_x - Kx(1 - 4x^2)(1 - y^2 + 2y^4) \\ y + \Omega_y + Kx(1 - 4x^2) \end{bmatrix}. \tag{5.2}$$

They generalized the Fibonacci recurrence by using

$$f_{n+3} = f_{n+2} + f_{n+1} + f_n, \qquad f_0 = f_1 = 0, f_2 = 1, \qquad n = 0, 1, 2, \ldots \tag{5.3}$$

on the grounds that it is 'the simplest possible ternary continued fraction expansion'. On the other hand, Kim and Ostlund [Kim and Ostlund, 1985; Kim and Ostlund, 1986] used

$$f_{n+3} = f_{n+1} + f_n, \qquad f_0 = f_1 = 0, f_2 = 1, \qquad n = 0, 1, 2, \ldots. \tag{5.4}$$

These authors looked for scaling by examining the eigenvalues of a two-step scaling matrix $\mathrm{diag}(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix})$ defined by solving

$$\begin{bmatrix} \Delta\Omega_x(n-1) \\ \Delta\Omega_y(n-1) \\ \Delta\Omega_x(n-2) \\ \Delta\Omega_y(n-2) \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \begin{bmatrix} \Delta\Omega_x(n) \\ \Delta\Omega_y(n) \\ \Delta\Omega_x(n-1) \\ \Delta\Omega_y(n-1) \end{bmatrix}$$

for $\{a, b, c, d\}$, where

$$\Delta\Omega_x(n) \equiv \Omega_x(n) - \Omega_x(n-1) \qquad n = 2, 3, 4, \ldots$$

etc., and $(\Omega_x(n), \Omega_y(n))$ are the parameter values for which the orbit of $(0,0)$ has rotation vector $(f_{n-1}, f_{n-2})/f_n$ [1]. None of these authors found critical scaling; that is, with $K = 1$, so that the map is noninvertible in a neighbourhood of zero. This is hardly surprising as the recurrences (5.3) and (5.4) have no theoretical justification, apart from an appeal to the imprecise concept of the 'most irrational pair'.

Note, however, that Hu, Mao, Kim and Ostlund did not consider *best* approximants. $(P_1/Q, P_2/Q)$ is said to be a best approximant to (ϕ_1, ϕ_2) if

$$\max(|q\phi_1 - p_1|, |q\phi_2 - p_2|) > \max(|Q\phi_1 - P_1|, |Q\phi_2 - P_2|)$$

whenever $p_1 < P_1, p_2 < P_2, q < Q$. Since one-dimensional continued fractions do produce best approximants, I decided to see if a similar idea could be used in two dimensions.

5.2   Simultaneous rational approximation

There is a vast literature on simultaneous rational approximation ([Szekeres, 1970] and references therein). In this literature, cubic number fields play a central rôle. A *cubic number field* [Cohen, 1993] is a set

$$\{ a_0 + a_1\zeta + a_2\zeta^2 \mid a_0, a_1, a_2 \in \mathbb{Q} \},$$

where ζ is a root of a monic irreducible cubic polynomial with integer coefficients. Every cubic number field K has a *discriminant* denoted $d(K)$, and the discriminant of the defining cubic (that is, $[(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)]^2$, where z_1, z_2, z_3 are the roots of the cubic) is always a squared-integer multiple of $d(K)$. The cubic number field is called *cyclic* if the discriminant of its defining cubic polynomial is a square. Thus the linear recurrence (5.3) has characteristic polynomial

$$z^3 - z^2 - z - 1$$

which has discriminant -44 and generates a field with the same discriminant, whereas the recurrence (5.4) has characteristic polynomial

$$z^3 - z - 1$$

---

[1] Note that in [Hu and Mao, 1987], n and $n - 1$ are reversed in equation (3.49), and that Hu and Mao do not explicitly state that they use a two-step procedure as above.

which has discriminant -23, again the discriminant of the generated cubic number field. The number 23 is in fact the smallest possible absolute value of the discriminant of any cubic number field, which is an intriguing analog of the fact that the golden mean in one dimension is associated with the quadratic number field $\mathbb{Q}(\sqrt{5})$ of smallest discriminant, namely 5. In fact, all these authors neglected to consider another cubic of small discriminant:

$$z^3 - z^2 - 1$$

which generates a field with $d(K) = -31$. In any case, all these polynomials have negative discriminant, a property not shared by the quadratic number field $\mathbb{Q}(\sqrt{5})$, which suggests that the analogy mentioned above is inappropriate.

Now, the smallest possible *positive* discriminant of a cubic number field is 49, corresponding to the cyclic cubic field $\mathbb{Q}(\theta)$ of $z^3 + z^2 - 2z - 1$, which has the nicely symmetric roots $\{\theta_1 = 2\cos(2\pi/7), \theta_2 = 2\cos(4\pi/7), \theta_3 = 2\cos(6\pi/7)\}$. (Note that $2\cos(2\pi/5) = (\sqrt{5} - 1)/2$, a very suggestive analogy!) It has also been shown by Szekeres [Szekeres, 1970; Szekeres, 1984; Szekeres, 1985; Szekeres, 1986] that this cubic number field plays a fundamental (though still incompletely understood) rôle in two-dimensional simultaneous rational approximant theory. On this basis, I decided to investigate further this case.

The first way to use the field $\mathbb{Q}(\theta)$ (I will not consider best rational approximants yet) is to generate rational approximants by

$$
\begin{aligned}
q(n+1) &= q(n) + 2p_2(n) - p_1(n) & (5.5)\\
p_1(n+1) &= p_2(n)\\
p_2(n+1) &= q(n)
\end{aligned}
$$

(with, for example, $p_1(1) = 0$, $p_2(1) = 1$, $q(1) = 2$ for $n = 1, 2, 3, \ldots$) and to use the ratios $(p_1(n), p_2(n))/q(n)$ as rotation vectors in a map such as (5.2). These ratios converge to $1/\theta_3^2$ and $1/\theta_3$ respectively, where $\theta_3 \approx -1.8019$ is the dominant eigenvalue of the characteristic polynomial of equation (5.5).

When I did this, computing the eigenvalues of the two-step scaling matrix at each stage, I obtained the results shown in Table 5.1. I regard this as strong evidence for scaling, since the eigenvalue $\lambda \approx -1.445 \approx \theta_3/\theta_1$ is present at all stages. If I am correct, this is the first reported evidence for critical scaling in two-torus maps.

For the second way to use the field $\mathbb{Q}(\theta)$, I wish to consider best simultaneous rational approximants to $\{\phi_1, \phi_2\}$, with ϕ_1, ϕ_2 irrational. Of course, the recurrences

p_1	p_2	q	Ω_x	Ω_y	λ_1	λ_2
75	136	244	0.33788994797	0.52471713785	-5.41338	-1.40636
136	244	441	0.33842876238	0.52104500234	1.31505	-1.44095
244	441	793	0.33817843567	0.52349504966	0.28819	-1.45572
441	793	1431	0.33836421868	0.52180021616	8.92446	-1.45301
793	1431	2576	0.33828561771	0.52294553760	0.26780	-1.45083
1431	2576	4645	0.33833719455	0.52215412112	-17.5783	-1.44425
2576	4645	8366	0.33829703996	0.52270526485	0.64748	-1.44369
4645	8366	15080	0.33832234497	0.52232670069	1.84701	-1.44709
8366	15080	27167	0.33830410806	0.52258993951	3.58308	-1.44319

Table 5.1: Evidence for scaling in the Hu-Mao torus map

used so far do not produce best approximants. I consider only pairs in the field of $z^3 + z^2 - 2z - 1$. I consider three separate choices of ϕ_1, ϕ_2:

1. $\phi_1 = \theta_1 - 1 \approx 0.246, \quad \phi_2 = -1 - \theta_3 \approx 0.801$

2. $\phi_1 = -\theta_2 \approx 0.445, \quad \phi_2 = -1 - \theta_3$

3. $\phi_1 = \theta_1 - 1, \quad \phi_2 = -\theta_2$

Now consider three corresponding systems of linear recurrence relations with periodic coefficients:

1.

$$r(k) = \begin{cases} 4r(k-1) + r(k-2) - r(k-3) & \text{if } k \equiv 0 \bmod 3 \\ r(k-1) + r(k-2) & \text{if } k \equiv 1 \bmod 3 \\ 4r(k-1) + r(k-4) & \text{if } k \equiv 2 \bmod 3 \end{cases} \quad k = 3, 4, 5, \ldots$$

2.

$$r(k) = \begin{cases} r(k-1) + r(k-2) - r(k-3) & \text{if } k \equiv 0 \bmod 6 \\ r(k-2) + r(k-3) & \text{if } k \equiv 1 \bmod 6 \\ 2r(k-1) + r(k-2) - r(k-3) & \text{if } k \equiv 2 \bmod 6 \\ r(k-1) + r(k-2) + r(k-3) & \text{if } k \equiv 3 \bmod 6 \\ 2r(k-1) + r(k-3) & \text{if } k \equiv 4 \bmod 6 \\ r(k-1) + r(k-3) & \text{if } k \equiv 5 \bmod 6 \end{cases} \quad k = 6, 7, 8, \ldots$$

3.

$$r(k) = \begin{cases} r(k-1) + r(k-3) & \text{if } k \equiv 0 \bmod 5 \\ r(k-1) + r(k-3) & \text{if } k \equiv 1 \bmod 5 \\ r(k-1) + 2r(k-2) + r(k-3) & \text{if } k \equiv 2 \bmod 5 \\ r(k-1) + r(k-3) & \text{if } k \equiv 3 \bmod 5 \\ 3r(k-1) + r(k-3) & \text{if } k \equiv 4 \bmod 5 \end{cases} \right\} k = 5, 6, 7, \ldots$$

In each of the three cases, consider three parallel sequences $p_1(k), p_2(k), q(k)$ which each satisfy the respective recurrence relation with respective initial values:

1. $p_1(0) = 0, p_1(1) = 0, p_1(2) = 1,$
   $p_2(0) = 0, p_2(1) = 1, p_2(2) = 3,$
   $q(0) = 1, q(1) = 1, q(2) = 4.$

2. $p_1(1) = 1, p_1(2) = 2, p_1(3) = 3, p_1(4) = 7, p_1(5) = 9,$
   $p_2(1) = 0, p_2(2) = 1, p_2(3) = 2, p_2(4) = 4, p_2(5) = 5,$
   $q(1) = 1, q(2) = 2, q(3) = 4, q(4) = 9, q(5) = 11.$

3. $p_1(1) = 0, p_1(2) = 1, p_1(3) = 1, p_1(4) = 3,$
   $p_2(1) = 0, p_2(2) = 1, p_2(3) = 2, p_2(4) = 6,$
   $q(1) = 1, q(2) = 3, q(3) = 4, q(4) = 13.$

In each case, the sequence of rational pairs $(p_1(k)/q(k), p_2(k)/q(k))$, are *precisely* the sequence of best rational approximants, as far as

1: $p_1(27) = 120860119535, p_2(27) = 392430343033, q(27) = 489352633642,$

2: $p_1(52) = 217782410144, p_2(52) = 120860119535, q(52) = 271570223498,$

3: $p_1(45) = 120860119535, p_2(45) = 217782410144, q(45) = 489352633642.$

Furthermore, these recurrences are unique subject to the requirement that only the previous three approximants are used if possible, and if this is not possible (in case 1 only), the coefficient of $r(k-4)$ be unity. I have confirmed this claim by explicit numerical computation of the best rational approximants up to about $q = 10^{10}$, in all three cases. When the rational approximant generated by my recurrences fails to be best, it is only marginally so, and I believe they continue to generate good approximants, and probably some of the best, after the critical k. I believe that these observations are new.

I speculate that the reason for this behaviour is the fact that the cubic $z^3 - 20z^2 - 9z - 1$ generates a field isomorphic to that of $z^3 + z^2 - 2z - 1$, but has one root about 20.44, and the other two roots of modulus less than 0.23. Thus, the asymptotics of the recurrence is rapidly dominated by the largest root. This is supported by the following observation: for case 2, I find that for $k = 9, 10, 11, \ldots$:

$$\begin{bmatrix} p_1(k-6) & p_2(k-6) & q(k-6) \\ p_1(k-7) & p_2(k-7) & q(k-7) \\ p_1(k-8) & p_2(k-8) & q(k-8) \end{bmatrix} \begin{bmatrix} 7 & 4 & 9 \\ 4 & 2 & 5 \\ 9 & 5 & 11 \end{bmatrix} = \begin{bmatrix} p_1(k) & p_2(k) & q(k) \\ p_1(k-1) & p_2(k-1) & q(k-1) \\ p_1(k-2) & p_2(k-2) & q(k-2) \end{bmatrix}.$$

The characteristic polynomial of the constant symmetric matrix C in the above equation is $z^3 - 20z^2 - 9z - 1$, and the eigenvalues of C are $\lambda_1 \approx 20.4426, \lambda_2 \approx -0.229521, \lambda_3 \approx -0.213128$. The growth rates of the denominators are defined by

$$g \equiv \liminf_{j \to \infty} q(j)^{1/j},$$

and are a measure of 'irrationality' of the pair. Figure 5.1 gives an example of the approach of $q(j)^{1/j}$ to its limit in case 1. The other two cases are similar. My numerical estimates of the respective lim infs are:

1. $g = 2.73$

2. $g = 1.65$

3. $g = 1.83$,

whereas from the largest eigenvalue of the matrix C: $\left(|\lambda_1|^{1/\text{period}}\right)$

1. $g = 2.734297237$

2. $g = 1.653571056$

3. $g = 1.828552501$.

Unfortunately, these recurrences generate approximants with a rapid growth of denominators, and I was unable to overcome the numerical difficulties of finding orbits of two-dimensional maps with the resulting large values of q in order to test

for scaling. However, I believe that there is potential for further investigation of these ideas.

I also considered using these results to generalize the Manton-Nauenberg scaling (4.4) to maps of $\mathbb{C}^2$. For this, we need to first generalize the Siegel linearization theorem.

5.3   The two-dimensional Siegel linearization theorem

Fornæss ([Fornæss, 1996], page 20, Theorem 4.6) shows the following theorem for $\mathbb{P}^n$=complex projective n-space (which for our purpose may be identified with $\mathbb{C}^2$ plus a point at infinity):

Theorem 2 *Let* $f : \mathbb{P}^n \to \mathbb{P}^n$ *be a holomorphic map of degree at least two and suppose that* Ω *is a Siegel domain. Then the boundary of* Ω *is contained in* $\overline{\cup_{j \geqslant 0} f^{<j>}(C)}$.

Here C is the critical set, that is, the set of points where the Jacobian of f is singular.

The relevant generalization of Siegel's linearization theorem has been given by Zehnder [Zehnder, 1977]:

Theorem 3 *Let* $z \mapsto f(z) = \Lambda z + \widehat{f}(z)$ *be a holomorphic map in a neighbourhood of zero in* $\mathbb{C}^n$, $\widehat{f}$ *contains only terms of order* $\geqslant 2$. *Assume* $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ *to be diagonal, the eigenvalues* $\lambda_k, 1 \leqslant k \leqslant n$ *satisfying [the diophantine conditions]*

$$|\lambda^j - \lambda_k| > C_0 |j|^{-\nu}$$

for all integer vectors $j = (j_1, \dots, j_n), j \geqslant 0$ *with* $|j| = \sum_{k=1}^{n} j_k > 1$. C_0 *and* ν *are two positive constants, and* λ^j *stands for* $\lambda_1^{j_1} \lambda_2^{j_2} \dots \lambda_n^{j_n}$.

Then there is a (unique) holomorphic map $z = u(\zeta) = z + \widehat{u}(\zeta)$ *in a neighbourhood of zero,* $\widehat{u}$ *containing terms of order* $\geqslant 2$ *only, such that*

$$f(u(\zeta)) = u(\Lambda \zeta).$$

Thus, we may consider the case $n = 2$ and maps such as

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto f\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = \begin{bmatrix} \lambda_1 z_1 + \mathcal{O}(2) \\ \lambda_2 z_2 + \mathcal{O}(2) \end{bmatrix} \tag{5.6}$$

where $\mathcal{O}(2)$ represents terms quadratic and higher in z_1, z_2, and $|\lambda_1| = |\lambda_2| = 1$. We then know that if the λs satisfy the diophantine conditions, the Schröder equation $f(u(\zeta)) = u(\Lambda\zeta)$ has a solution. We also know that the boundary of the Siegel domain is contained in the closure of the forward orbit of the critical set.

I believe that my best-approximation recurrences could be used in this context to look for scaling, but it is not clear to me how to analyse the orbits, and at the time of writing I have no definite results. However, I believe that there is considerable potential for future research here.

5.4 A torus map based on Jacobi's sn

Finally in this chapter, I discuss the 'Mandelbrot' set and Julia sets arising from the iteration of a map of the two-torus defined in terms of Jacobi's elliptic function sn. The idea here is to see if the requirement of complex analyticity produces any results different to the $\mathbb{R}^2$-analytic case considered in the previous section.

I consider the torus map defined by the complex analytic family $f_\Omega(z) \equiv \Omega + z - \text{sn}(\gamma z, m)/\gamma$. Here sn is Jacobi's elliptic function, which is doubly periodic and has two poles in each period rectangle [Abramowitz and Stegun, 1965]. Ω is a complex parameter, and m and γ are real constants. With the choices $m = (3 - 2\sqrt{2})^2$ and $\gamma = 4K(m)$, where K is the complete elliptic integral of the first kind [Abramowitz and Stegun, 1965], we have the relation $f_\Omega(z + k + il) = f_\Omega(z) + k + il$ for integer k, l and for all z, Ω. Thus this is a natural generalization of the sine circle map family. Note also that f_Ω is real whenever z and Ω are real. Thus this family can be considered as a complex perturbation of the real sine map, as a nonzero imaginary part is introduced to Ω. If we desire such a complex analytic extension, an unavoidable consequence is the presence of poles, in this case at $1/2$ and $(1+i)/2$ in each unit square (which is the period rectangle of sn in the complex plane).

5.4.1 Numerical results

The function sn can be easily computed with the standard arithmetic-geometric mean iteration [Borwein and Borwein, 1986]. My first study was of the parameter space of the family f_Ω. Due to the symmetries of the function ($f_{1-\Omega}(0) = 1 - f_\Omega(0)$ and $f_{i-\Omega}(0) = i - f_\Omega(0)$), it is sufficient to study the region $0 < \Re\Omega, \Im\Omega < 1/2$. For a large number of values of Ω in this region, I computed the orbit of zero under the map f_Ω, that is, the sequence $\{0, f_\Omega(0), f_\Omega(f_\Omega(0)) \equiv f_\Omega^{<2>}(0), \ldots\}$. If

convergence to a quasiperiodic orbit was detected (that is, convergence to a point z_0 satisfying $f_\Omega^{<q>}(z_0) = z_0 + k + il$ for some integers q, k, l), then the point Ω was coloured black. The result is shown in Figure 5.2. This can be considered the Mandelbrot set for the family f_Ω. The 'ears' are actually connected to the main regions; the gaps are an artifact of the algorithm used. The following pseudocode outlines the method I used:

```
compute m and gamma
for j from 0 to 1000 do
  for k from 0 to 1000 do
     omega:=(j+ik)/2000
     z:=0
     for n from 1 to 1000 do # preiterate
        z:=omega+z-sn(gamma*z,m)/gamma
     endfor
     z0:=z
     for n from 1 to 20 do
        z:=omega+z-sn(gamma*z,m)/gamma
        if both Re(z-z0) and Im(z-z0) are within 1.0e-6 of integers
           then color pixel (j,k) black
     endfor
  endfor
endfor
```

Note that each individual component is topologically just a Mandelbrot set for the cubic family $z^3 + c$. This can be understood from the fact that the family f_Ω has a cubic critical point at the origin. Each such component has constant 'period' q, and varying k and l, depending on the bifurcation path followed from the central region. That is, each complete component, including 'ears', has constant q.

5.4.2   Scaling

Since one of my aims was to look for possible scaling laws in the family f_Ω, I will first briefly recall the standard results for the sine family (see Chapter 4) Consider the sequence of rationals $\{p_i/p_{i+1}\} = \{1/1, 1/2, 2/3, 3/5, \ldots\}$, so that p_i is the ith Fibonacci number, and the limit of the sequence is the golden mean $(\sqrt{5} - 1)/2$.
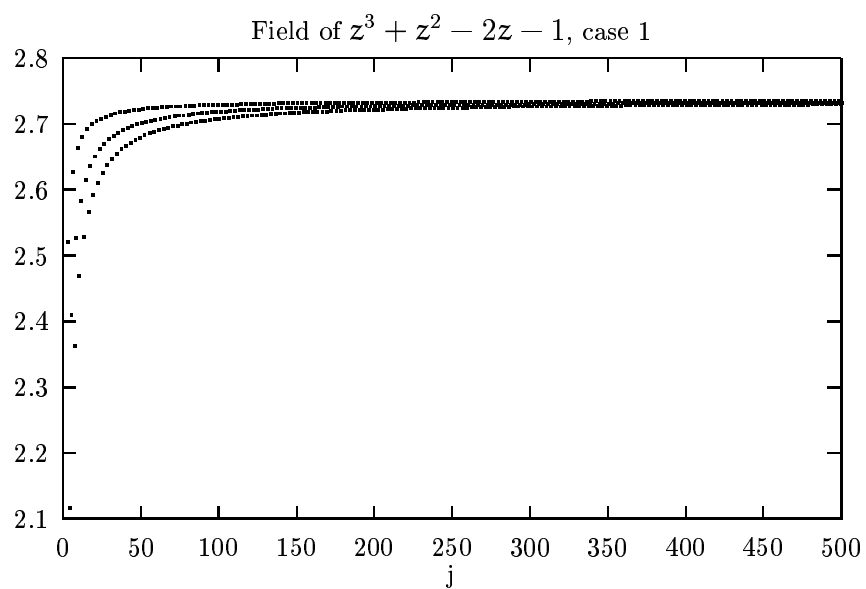
Figure 5.1: The approach of $q(j)^{1/j}$ to its limit
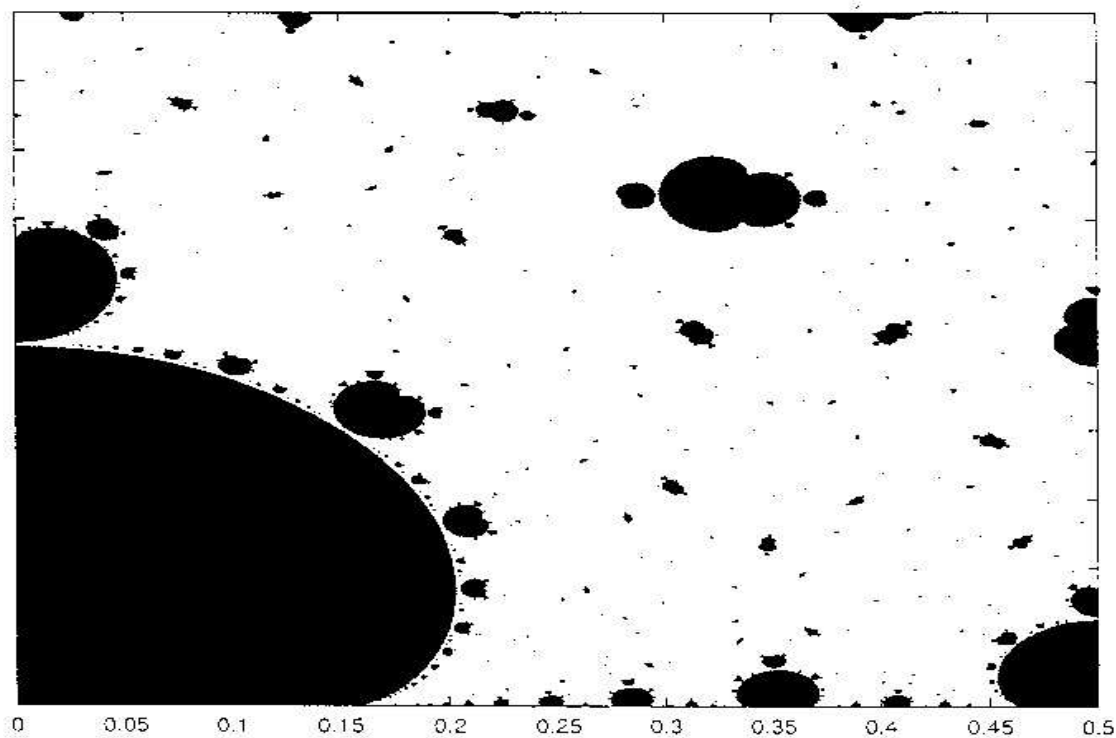
Figure 5.2: The mandelset of $f_\Omega(z) = \Omega + z - \text{sn}(\gamma z, m)/\gamma$

For each i, let Ω_i be the smallest positive value of Ω such that $f_{\Omega_i}^{<p_{i+1}>}(0) = p_i$. Then the map f_{Ω_i} has winding number p_i/p_{i+1}. In other words, the point zero is just shifted by p_i units under p_{i+1} iterations of the map; or, modulo the integer lattice, it is a fixed point of $f^{<p_{i+1}>}$. Then it is found that $\lim_{i\to\infty} \Omega_i/\Omega_{i+1}$ exists and takes the value $\delta \approx -2.833$ (see Chapter 4). I now consider the complex sn case, and of course the winding number, similarly defined, becomes complex.

First of all, within each connected black region of Figure 5.2, we have the well-understood scaling behaviour of complex cubics described in § 3.1. For example, the largest region, centered on the origin, has winding number $(0 + 0i)/1$, and moving up the imaginary axis we have a bifurcation to a winding number $(0 + 0i)/2$.

The other regions are (in decreasing order of size) $(0 + 0i)/3$, $(0 + 0i)/4$, etc. Secondly, starting at $\Omega = 1/2$, we have regions of winding number $(1 + 0i)/3$, $(1 + 0i)/4$, $(1 + 0i)/5$ and so on in decreasing order of size along the real axis. At approximately $\Omega = 1/2 + 0.27i$, the region has winding number $(1 + i)/4$. To produce an analog of the Fibonacci scaling law for the real sine map, we would need to consider one of the two-dimensional simultaneous approximation algorithms discussed in § 5.2. Then we would need to locate regions corresponding to a sequence of approximants with rational real and imaginary parts. However, I found the numerical problem of finding the appropriate parameter values too difficult, and at this stage, I have no scaling results for this type of map.

5.4.3   Julia sets

By fixing the parameter Ω, one may study the orbits of the map f_Ω as a function of the initial point of the orbit. Figure 5.3 was computed by colouring black all those initial points for which the orbit under f_0 does not exceed a modulus of 10^3 in 5000 iterations. It can be considered the filled Julia set of f_0, although other definitions [Bergweiler, 1993], perhaps more appropriate for maps such as these, have been proposed.
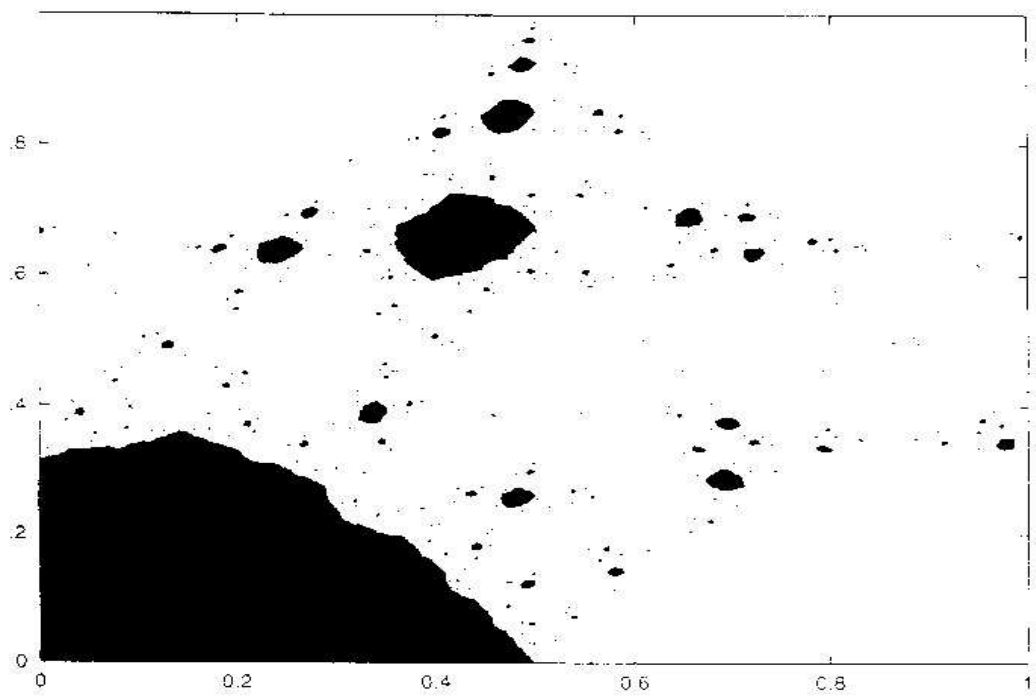
Figure 5.3: The Julia set of $f_0(z) = z - \mathrm{sn}(\gamma z, m)/\gamma$

Chapter 6

Quaternion maps

Adriano de Armado: *'Is that one of the four complexions?'*

Love's Labour's Lost, act 1, scene 2

6.1   Introduction

Complex analytic dynamical systems are well understood [Devaney, 1989; Milnor, 1990], and the corresponding Mandelbrot and Julia sets have created much interest [Peitgen and Richter, 1986]. It is therefore natural to ask if these concepts have an analog in higher dimensions. Indeed [Norton, 1982] and [Pickover, 1990] have drawn what they claim are the quaternion analogs of the complex Mandelbrot and Julia sets. However, as I will make clear, these authors considered a very restricted class of maps, apparently without realizing that their maps were essentially equivalent to complex maps. In this chapter I will introduce a general theory of quaternion iteration.

In particular, I wish to examine the question whether there exist any generic structures playing a central rôle, analogous to the ordinary Mandelbrot set in the complex case, and thus whether there is a Feigenbaum scaling theory, distinct from that described in the earlier chapters of this thesis.

Although some aspects of quaternion iteration theory have been discussed [Holbrook, 1987; Kozak and Petek, 1992; Petek, 1992; Heidrich, 1994; Bedding and Briggs, 1996a], there is no systematic description in the literature. I therefore decided to present here a fairly full exposition of the background and the results I obtained.

Quaternions are an extension of the idea of complex numbers to four dimensions, and I begin by discussing the iteration of linear and quadratic functions of the quaternions, and examine the rôle played by regularity (the analog of complex analyticity) in this context. In contrast to the complex case, regularity is not automatically preserved by composition of quaternion functions. I find that demanding preservation of regularity is in fact too restrictive, yielding very little new beyond the complex case. Finally, the quaternion generalization of the Mandelbrot set is described.

6.2   Quaternions

I denote Hamilton's quaternions by $\mathbb{H}$, that is, the set $\{q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, q_m \in \mathbb{R}\}$ with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. $\mathbb{H}$ is an associative, but non-commutative, division algebra. Each subspace of real dimension two (for example, the 1-$\mathbf{i}$ plane) is isomorphic to $\mathbb{C}$. For an excellent treatment of the algebraic properties of $\mathbb{H}$, see [Koecher and Remmert, 1990].

The subset of pure quaternions (those with zero real part) is denoted $\mathrm{Pu}(\mathbb{H})$. The pure part of q is the 3-vector corresponding to the $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ components, and is denoted $\mathrm{Pu}(q)$. The conjugation operation $q \mapsto \overline{q}$ negates the pure part of q. If $a \in \mathbb{H}$ is a unit quaternion (that is, $a\overline{a} = 1$), then $aq\overline{a}$ is a rotation of the pure part of q. As an explicit example, if $c = \cos(\psi/2)$ and $s = \sin(\psi/2)$, then

$$(c + s\mathbf{i})q(c - s\mathbf{i}) = q_0 + q_1\mathbf{i} + (q_2\cos\psi - q_3\sin\psi)\mathbf{j} + (q_2\sin\psi + q_3\cos\psi)\mathbf{k}.$$

This is a rotation through an angle ψ about the $\mathbf{i}$ axis. Clearly the choice $\psi = \arctan(-q_3/q_2)$ annihilates the $\mathbf{k}$ component of the result. In fact, for any $q \in \mathbb{H}$ we can always find an a such that $aq\overline{a}$ is purely complex. I will write $\phi_a(q) \equiv aq\overline{a}$, and $\phi_a^{<-1>}$ for the inverse of this operator.

6.3   Quadratic quaternion maps

Consider the map $f_c(q) \equiv c + q^2$. Then

$$\phi_a^{<-1>} \circ f_c \circ \phi_a(q) = \overline{a}ca + q^2$$

so that for each c, an a can be found so that $\overline{a}ca$ is purely complex. Then the orbit with seed zero will be purely complex, and the parameter dependence of

the dynamics will be described by the ordinary mandelset $M(c + z^2, 0)$. This is the set of c values such that the orbit of the origin is bounded under the map $z \mapsto c + z^2$. In other words, the quaternion quadratic map, when restricted to the orbit of zero, is conjugate to a complex quadratic map (that is, equivalent under an invertible change of variables). The value of a required to achieve this will be an $\mathbb{R}^4$-analytic function of c. This means that if we define the *quandelset* to be the set of c values such that the orbit of zero is bounded under $c + q^2$, then locally (for example, in some neighbourhood of an n-furcation point) the quandelset is obtained from the mandelset by a rotation, smoothly dependent on the parameter c. In particular, the slices in the 1-**i**,1-**j** and 1-**k** planes are exactly the same as the ordinary complex mandelset. Furthermore, for each c, the entire orbit of the origin lies in the subspace of $\mathbb{H}$ containing 0,c and c^2.

Next I investigate the stability of the quaternion n-cycles. The stability is determined by the eigenvalues of the product of Jacobian matrices:

$$J_{f_c}(0)J_{f_c}(f_c(0)) \cdots J_{f_c}(f_c^{<n-1>}(0)).$$

Now for $f_c(q) = c + q^2$, with $c \in \mathbb{C}$, we have

$$J_{f_c}(q) = 2 \begin{bmatrix} q_0 & -q_1 & 0 & 0 \\ q_1 & q_0 & 0 & 0 \\ 0 & 0 & q_0 & 0 \\ 0 & 0 & 0 & q_0 \end{bmatrix},$$

so that the eigenvalues of $J_{f_c}(q)$ are $2q_0$ and $2(q_0 \pm iq_1)$. Because of the block structure of $J_{f_c}(q)$, these eigenvalues are multiplicative along an orbit, and we have exact equivalence to the complex case where the eigenvalue of $J_f(z) = f'(z) = 2z$ is $2z$. The additional eigenvalue of $2q_0$ does not affect the stability, because its modulus is always less than or equal to $2|(q_0 \pm iq_1)|$.

Now the ordinary complex mandelset contains hyperbolic components on which the map has a stable cycle of constant period. These hyperbolic components are each conformal to the unit disk, and meet tangentially at n-furcation points. In the quandelset, these components are now bounded by surfaces, the topology of which depends on whether the original hyperbolic component intersects the real axis. For example, the period-one region becomes a topological three-sphere. Because of the uniqueness of the rotation ϕ_a, these components never intersect. For example, the two period-three components attached to the main cardioid of the mandelset are joined in the quandelset by 'tubes' in the two additional dimensions. Also, the

self-similar features, quantified by the scaling constants known as feigenvalues are just the usual complex ones described in § 3.1, since these are invariant under an analytic re-parameterization.

Note that this entire argument also works for the maps $1 + cq^2, 1 + qcq$ etc. One might ask why a similar argument cannot be used to reduce the complex case to the real case. The answer is that a rotation in complex space is of the form $\phi_a(z) = az$, where $|a| = 1$. Thus if $f_c(z) = c + z^2$, then

$$\phi_a^{<-1>} \circ f_c \circ \phi_a(z) = a^{-1}c + az^2.$$

We could therefore make $a^{-1}c$ real, but the factor a remains in front of z^2, and the orbit of zero does not remain real. It is the lack of commutativity that makes the argument work in the quaternion case, whereas in the complex case the rotation commutes with the dynamics.

This completes my discussion of the map $c + q^2$. I conclude that there is no new interesting dynamics in this map. Now, this map does *not* share with its complex analog the property of analyticity, a property known to be crucial for the existence of mandelsets with the topology described above. Note that if we allow any quaternion coefficients in the quadratic map, we can in fact produce an arbitrary quadratic map of $\mathbb{R}^4$ [Sudbery, 1979]. Thus some restrictions are required if we are to produce any properties characteristic of quaternions. Clearly the map $c + q^2$ does not play any fundamental rôle analogous to that of the map $c + z^2$ in the complex case. Indeed, even in the non-analytic complex case, more general maps (depending on $\overline{z}$ as well as z) can exhibit other types of bifurcations than the period n-tupling observed in the mandelset. In the next section I will consider quaternion analogs of the concept of analyticity.

6.4   Regular and regularly iterable maps

The property corresponding to complex analyticity for quaternion (or, more generally, Clifford algebra) functions is known as regularity or monogenicity, and is a requirement that the function satisfy a generalized Cauchy-Riemann equation. This concept originated with Fueter[1] in the 1930s [Fueter, 1932; Fueter, 1934; Fueter,

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[1] Since Fueter is little known, I add the following information from [Bieberbach, 1931]: Rudolf F. Fueter was born in Basel in 1880, studied in Göttingen and Marburg, and was professor in Basel, Karlsruhe and in Zürich from 1917, where he did his work on quaternions.

1935/6; Fueter, 1937*a*; Fueter, 1937*b*; Fueter, 1938] and has recently been investigated further [Sudbery, 1979; Brackx, Delanghe and Sommen, 1983; Gilbert and Murray, 1991].

Following these authors, I define the left Cauchy-Riemann-Fueter operator by

$$\partial \equiv 2\overline{\partial}_l \equiv \frac{\partial}{\partial q_0} + i\frac{\partial}{\partial q_1} + j\frac{\partial}{\partial q_2} + k\frac{\partial}{\partial q_3},$$

so that the condition for f to be left-regular at q is

$$\partial f(q) = 0. \tag{6.1}$$

A similar definition may be made for right-regularity.

The above definition proves very useful in the study of quaternion integral theorems and generalization of Cauchy's theorem, but suffers from the drawback that the common functions of interest, such as polynomials and even the identity function itself, are not left-regular or right-regular functions.

For this reason, I mention an alternative definition due to Rinehart [Rinehart, 1960] and Cullen [Cullen, 1965]. We define the *Cullen differential operator* by

$$\partial_C \equiv \frac{1}{2}\left(\frac{\partial}{\partial q_0} + \frac{Pu(q)}{r}\frac{\partial}{\partial r}\right),$$

where $r^2 \equiv q_1^2 + q_2^2 + q_3^2$. A function $f : q \mapsto f(q)$ can now be defined as *Cullen-regular* at q if it satisfies

$$\partial_C f(q) = 0.$$

The variable r is a radial distance on a two-sphere in the pure part of $\mathbb{H}$ and $Pu(q)/r$ is then a Cullen-regular unit radial vector. ∂_C is a generalization of the usual complex differential operator $\partial/\partial\overline{z}$. If we restrict q to any of the planes in which some pair of q_1, q_2, q_3 vanishes, then $\partial_C, \overline{\partial_C}$ are precisely the complex differential operators in that plane; thus it should not be surprising that, as in the complex case, the Cullen definition of regularity includes all (real coefficient) polynomial functions of q as regular functions. This result is easy to see when it is realized that such polynomials always take the form

$$p(q) = f(q_0, r) + g(q_0, r)\frac{Pu(q)}{r}.$$

Since $Pu(q)/r$ commutes with ∂_C, the regularity follows by recognizing that the situation is identical to the case of Cauchy-Riemann equations for a complex polynomial.

As well as polynomial functions, Cullen-regular functions include rational functions with real coefficients, and all of the elementary transcendental functions (defined by their usual Taylor expansions) such as the exponential, logarithmic, and trigonometric functions. With the above definitions, the usual procedures for real or complex differentiation carry through, so that we have

$$
\begin{aligned}
\partial_C q &= 0 \\
\overline{\partial_C} q &= 1 \\
\partial_C (q^n) &= 0 \\
\overline{\partial_C} (q^n) &= n q^{n-1} \\
\partial_C (f(q)) &= 0.
\end{aligned}
$$

I will make no further use of the concept of Cullen-regularity, since when applied to iterates of a map it produces only a situation equivalent to complex analyticity. Thus use of the unqualified term 'regular' will henceforth refer to the definition of left-regularity in equation (6.1).

Although the concept of regularity is standard, the idea of regular iterability has not been considered before. By *regularly iterable*, I mean that all iterates $f^{<1>}, f^{<2>}, f^{<3>}, \ldots$ are regular. This (rather strong) restriction will need detailed investigation. I begin with regularly iterable linear maps.

6.5   Regularly iterable linear maps

I first consider the left-$\mathbb{H}$-linear map $f(q) = qa$, where a is a constant quaternion. We have $f'(q) = -4a$. Considered as a map on $\mathbb{R}^4$, the eigenvalues are $\mathfrak{R}(a) \pm i|\mathrm{Pu}(a)|$. So the origin is stable if $|f'(0)|^2 < 4$, or $4|a| < 1$. Note, however, that $\partial f(q) = -2a$ is zero iff $a = 0$. Thus, if we wish to consider non-trivial regular linear functions, we must drop the $\mathbb{H}$-linearity condition, and instead impose the requirement of real-homogeneity.

The most general left-regular map which is real-homogeneous of degree one is

$$
f(q) = (\mathbf{i}q + q\mathbf{i})a + (\mathbf{j}q + q\mathbf{j})b + (\mathbf{k}q + q\mathbf{k})c,
$$

where a, b, c are constant quaternions. Represented as a map on $\mathbb{R}^4$, this is

$$
f(q) = -2 \begin{bmatrix} a_1 + b_2 + c_3 & a_0 & b_0 & c_0 \\ -b_3 + c_2 - a_0 & a_1 & b_1 & c_1 \\ -c_1 + a_3 - b_0 & a_2 & b_2 & c_2 \\ -a_2 + b_1 - c_0 & a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \tag{6.2}
$$

where $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ (a_n real) etc. A sufficient condition for $f^{<k>}$ to be regular is that b and c are both real multiples of a. To see this, we make the substitution $b = xa$, $c = ya$, ($x, y \in \mathbb{R}$) in equation (6.2) obtaining

$$
f(q) = Aq \tag{6.3}
$$

with f and q now regarded as column vectors and

$$
A = -2 \begin{bmatrix} a_1 + xa_2 + ya_3 & a_0 & xa_0 & ya_0 \\ -xa_3 + ya_2 - a_0 & a_1 & xa_1 & ya_1 \\ -ya_1 + a_3 - xa_0 & a_2 & xa_2 & ya_2 \\ -a_2 + xa_1 - ya_0 & a_3 & xa_3 & ya_3 \end{bmatrix}.
$$

A detailed study of the iteration properties of maps of the type (6.3) will be now be undertaken. Because of the complexity of the calculations, I first summarize the results to be obtained: it will be shown that if the maps (6.3) are expressed in an appropriate basis, they are again conjugate to linear maps in the complex plane. As such, they lose much of their interest from a quaternion point of view. However, maps of type (6.3) are not the only regularly iterable linear maps. An example of a slightly different kind has been given, for example, in [Heidrich, 1994]. It is possible to write down a necessary and sufficient set of simultaneous multinomial equations for the components of the quaternions a, b, c to yield a regularly iterable map. Included in these equations is the condition that the determinant of the matrix A should be zero. In solving these equations, I have recovered the case (6.3) above as well as the solution mentioned in [Heidrich, 1994] and a multiplicity of other cases. Investigation of the eigenvalues of A for these many cases reveals that most reduce to complex maps in a two-dimensional subspace of $\mathbb{H}$. There are a few exceptional cases. In general, these cases must project an initial value of q into at most a three-dimensional subspace of $\mathbb{H}$ (because of the zero determinant). Of those investigated so far, none exhibits very interesting iteration behaviour. An example of the kind of behaviour found is one where the iterates move along a circular helix converging to the origin or diverging to infinity.

I now proceed to completely classify regularly iterable linear maps. While it is of greater interest to study quadratic and higher degree maps, I consider it essential to first understand what happens in the linear case. Analogous investigations have been undertaken in [Heidrich, 1994] for the far simpler case of the space $\mathbb{R}^3$. For this space, it was concluded in [Heidrich, 1994] that regular iterability constrains all linear maps to be equivalent to purely complex maps. It has been speculated [Heidrich, 1994], that a similar state of affairs probably exists in the full quaternion case. I show here that these speculations are almost, but not entirely, correct.

The demand that iteration preserve regularity is now just $\partial(f^{<n>}(q)) = 0$ for all iterates n, which may be viewed as a constraint on the matrix A given above, so that $\partial(A^n q) = 0$. To progress further, we need to examine properties of powers of the matrix A.

For a 4×4 matrix, the Cayley-Hamilton theorem implies that

$$A^4 = \alpha_1 A^3 + \alpha_2 A^2 + \alpha_3 A + \alpha_4 I_4,$$

where I_4 is the 4×4 unit matrix and the coefficients α_i are functions of the various trace invariants of A. In particular, $\alpha_4 = -\det(A)$. Higher powers of A than the fourth may be similarly expressed in terms of the four lowest non-negative integer powers, by repeated use of the Cayley-Hamilton formula.

Let us now assume that a regular linear function $f(q)$ is known also to have regular second and third iterates. Then for this function, we have $\partial(Aq) = \partial(A^2 q) = \partial(A^3 q) = 0$. Applying ∂ to A^4 and higher powers of A acting on q, then replacing these powers by their expressions in terms of lower powers on the right of the Cayley-Hamilton expressions shows us that regularity of all higher powers is achieved solely by demanding the vanishing of $\det(A)$. This is true because the three non-zero lower powers give regular maps, while $\partial(\det(A)I_4 q) = -2\det(A)$. The task is therefore reduced to that of finding all A for which the first three powers of A are regular maps, and the determinant of A is zero. This problem involves a significant amount of complicated algebra, for which the assistance of the commercial computer algebra program Maple was invoked. The next task is therefore to formulate the problem in a manner which is convenient for programming with Maple.

Let M be any 4×4 matrix with components m_{ij} acting on q regarded as a

vector in $\mathbb{R}^4$ and consider the result of acting on Mq with the derivative ∂:

$$\partial(Mq) = \begin{bmatrix} m_{00} - m_{11} - m_{22} - m_{33} \\ m_{10} + m_{01} + m_{32} - m_{23} \\ m_{20} + m_{02} + m_{13} - m_{31} \\ m_{30} + m_{03} + m_{21} - m_{12} \end{bmatrix}.$$

The general solution of $\partial(Mq) = 0$ is precisely $M = A$ as expected; however, we see that the special structure of A has been determined by four equations. In order to force the second and third iterates of f to be regular, we just need to rewrite the four equations substituting components of A^2 and A^3 for those of M, then impose the condition $\det(A) = 0$ to establish regular iterability. Given an already regular A, we have to solve nine multinomial equations for the twelve unknown real quantities appearing in A. I now proceed to write these equations in a manner suitable for input to Maple.

I first define four matrices η_i by

$$\eta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\eta_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\eta_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\eta_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The nine conditions for regular iterability of A may now be written as

$$\mathrm{Tr}(A^2\eta_i) = \mathrm{Tr}(A^3\eta_i) = \det(A) = 0, \qquad i = 0, 1, 2, 3.$$

Before attempting to solve the above equations, I will study transformations that preserve regularity. An understanding of this topic will turn out be indispensable in finding the most general solution sets. Sudbery [Sudbery, 1979] shows that the general conformal (that is, Möbius or angle-preserving) transformation does not preserve the regularity of a regular quaternion function. However, given a regular function f, the special combination

$$F(q) = \frac{1}{|\beta - \alpha\gamma^{-1}\delta|^2} \frac{(\gamma q + \delta)^{-1}}{|\gamma q + \delta|^2} f(\nu(q))$$

is regular, where $\nu(q) = (\alpha q + \beta)(\gamma q + \delta)^{-1}$ and $\alpha, \beta, \gamma, \delta$ are constant quaternions.

Since we are only interested in linear maps, we set $\gamma = \beta = 0$. Sudbery [Sudbery, 1979] then shows in his proposition 5(ii) that this special case preserves regularity. In other words, if f is regular, then so is $\delta^{-1}f(\alpha q\delta^{-1})$. (The constant real factors do not affect regularity.) Since we only care about left-regularity, we may also append any constant quaternion to the right hand side. I choose to multiply by α on the right. We now have a transformed regular linear map of the form $F(q) = \delta^{-1}A\alpha q\delta^{-1}\alpha$. Choosing $\delta = \bar{\alpha} = \alpha^{-1}$, we are left with $F(q) = \bar{\alpha}A\alpha q$ which is an orthogonal transformation on $Pu(\mathbb{H})$ [Koecher and Remmert, 1990]. From a purely practical point of view, however, I will later wish to regard the transformation as a symmetry of A, maintaining the regular iterability of the linear map Aq. In summary, if A is a regularly iterable linear map, then so also is MAM^{-1} where M is as given on page 219 of [Koecher and Remmert, 1990] as

$$M = \frac{1}{\phi} \begin{bmatrix} \phi & 0 & 0 & 0 \\ 0 & \kappa^2 + \lambda^2 - \mu^2 - \nu^2 & 2(\lambda\mu - \kappa\nu) & 2(\kappa\mu + \lambda\nu) \\ 0 & 2(\kappa\nu + \lambda\mu) & \kappa^2 - \lambda^2 + \mu^2 - \nu^2 & 2(\mu\nu - \kappa\lambda) \\ 0 & 2(\lambda\nu - \kappa\mu) & 2(\kappa\lambda + \mu\nu) & \kappa^2 - \lambda^2 - \mu^2 + \nu^2 \end{bmatrix}$$

with $\phi = \kappa^2 + \lambda^2 + \mu^2 + \nu^2$.

M is a matrix representation of a general element of an $SO(3)$ subgroup of the group $SO(1,3)$. It is clear why M leaves regular iterability intact when one examines its effect on the matrices $\eta, \eta_1, \eta_2, \eta_3$. We find $M\eta M^{-1} = \eta$ and further, η_1, η_2, η_3 form a triplet under the group rotation (that is,

$$M\eta_i M^{-1} = \sum_{j=1}^{3} a_{ij}\eta_j,$$

for some real a_{ij}). Since matrices can be cycled in a trace without altering its value, it is easy to demonstrate that the 13 defining conditions for regular iterability are

invariant under rotation of A with M. Later, I will use M to make a convenient choice for some components of A, thereby facilitating the solution of some of the trace equations.

6.5.1 Regularity of the second iterate

Every solution which follows could, in principle, give rise to other solutions by interchanging a, b and c. These alternative solutions are to be regarded as not new and so will be ignored. In terms of components of A, the four equations $\partial(A^2 q) = 0$ become

$$a_1 b_2 + a_1 c_3 + b_2 c_3 - b_1 a_2 - c_1 a_3 - c_2 b_3 = 0$$
$$-b_3 a_1 + c_2 a_1 + b_1 a_3 - c_1 a_2 = 0$$
$$b_1 c_2 - c_1 b_2 + a_3 b_2 - a_2 b_3 = 0$$
$$-c_1 b_3 + c_2 a_3 - a_2 c_3 + b_1 c_3 = 0.$$

As has already been noted one possible solution set for these equations is obtained by observation to be $\{b = xa, c = ya\}$, with $x, y \in \mathbb{R}$ and $a = (a_1, a_2, a_3)$ etc. This set will be referred to as $\mathcal{M}$.

When asked to find all solution sets, Maple produces 52 sets; however, many of these are special cases of the one above, while others contain non-real expressions for one or more of the components of a, b, c (which can be rejected). After careful examination of the solution sets, it is possible to determine that there is only one further independent set beyond that already given. I name this alternative set $\mathcal{N}$. It takes the following form:

$$a_2 = b_1, \ a_3 = c_1, \ b_3 = c_2, \ a_1 = (b_1^2 + c_1^2 + c_2^2 - b_2 c_3)/(b_2 + c_3)$$

with all other components independent at this stage. The sets $\mathcal{M}$ and $\mathcal{N}$ now constitute a complete solution to the problem of finding all twice regularly iterable linear quaternion maps. The solution sets $\mathcal{M}$ and $\mathcal{N}$ have been constructed to satisfy four of the nine desired constraints (though by coincidence, set $\mathcal{M}$ also results in a zero determinant for A). These sets must now be constrained further in order to satisfy regularity of the third and higher iterates.

6.5.2 Regularity of the third and higher iterates

Previously, it was noted that if the solution set $\mathcal{M}$ is supplemented with the further conditions $\{b_0 = xa_0, c_0 = ya_0\}$ the result is a fully regularly iterable map. It was

further shown that the resulting map is always equivalent to a complex linear map in some complex subspace of the quaternions, and thereby loses its interest in a quaternion context. Heidrich [Heidrich, 1994] has pointed out the existence of another form of regularly iterable linear map not of the kind discussed here. However, this new map also turns out to be equivalent to a complex map. These results led to the speculation that only complex-equivalent maps would be found. I now attempt to verify this speculation by finding all solution sets for all of the necessary constraints.

6.5.3   Solution set $\mathcal{M}$

Since matrices A of the type $\mathcal{M}$ already have zero determinant, Maple was instructed to substitute solution set $\mathcal{M}$ only into the four expressions for $\mathrm{Tr}(A^3\eta_i)$, (denoted henceforth by t_i) and simplify. The results are as follows:

$$
\begin{aligned}
t_0 &= -a_0{}^2 y a_3 - a_0{}^2 x a_2 + b_0 a_2 a_0 - c_0 a_2 a_1 - b_0 y a_1{}^2 + b_0 a_3{}^2 y - \\
&\quad b_0{}^2 y a_3 - c_0 a_2{}^2 x + c_0 x a_1{}^2 - c_0{}^2 x a_2 + b_0 a_3 a_1 + c_0 a_3 a_0 - \\
&\quad b_0{}^2 a_1 - c_0{}^2 a_1 - a_0 x^2 a_3 a_2 - a_0 x a_3{}^2 y + a_0 y a_2{}^2 x a_0 y^2 a_2 a_3 - \\
&\quad b_0 y a_1 x a_2 - b_0 y^2 a_1 a_3 + b_0 a_3 x a_2 - c_0 a_2 y a_3 + c_0 x^2 a_1 a_2 + \\
&\quad c_0 x a_1 y a_3 - a_0 x a_3 a_1 + a_0 y a_2 a_1 + a_0 x a_1 b_0 + c_0 x a_3 b_0 + \\
&\quad a_0 y a_1 c_0 + b_0 y a_2 c_0
\end{aligned}
$$

$$
\begin{aligned}
t_1 &= a_1 b_0 a_2 + a_0 y^2 a_2{}^2 + a_1 c_0 a_3 + a_0{}^2 x a_3 - a_0{}^2 y a_2 + a_0 c_0 a_2 + \\
&\quad a_0 x^2 a_3{}^2 - a_0 b_0 a_3 - a_0 a_1 x a_2 - a_0 a_1 y a_3 - 2 a_0 x a_2 y a_3 + a_0 b_0 y a_1 - \\
&\quad a_0 c_0 x a_1 + a_2 b_0 y a_3 + a_2 c_0 x a_3 - x a_1{}^2 b_0 - y a_1{}^2 c_0 - b_0 x a_3{}^2 - \\
&\quad c_0 y a_2{}^2 + y^2 a_1{}^2 a_0 + x^2 a_1{}^2 a_0
\end{aligned}
$$

$$
\begin{aligned}
t_2 &= b_0{}^2 y a_1 + b_0 c_0 a_2 - a_0 a_3{}^2 x + b_0 y^2 a_1{}^2 - b_0{}^2 a_3 - b_0 a_1 x a_2 - \\
&\quad 2 b_0 a_1 y a_3 - b_0 x a_2 y a_3 + b_0 a_0 x a_3 - b_0 a_0 y a_2 - b_0 c_0 x a_1 + x^2 a_1 a_0 a_2 + \\
&\quad x a_1 a_0 y a_3 + x a_1 c_0 a_3 + x^2 a_2 c_0 a_3 + b_0 a_3{}^2 + y^2 a_2{}^2 b_0 - a_2{}^2 a_0 x - \\
&\quad c_0 y a_2{}^2 x - y a_1{}^2 c_0 x + b_0 a_2{}^2
\end{aligned}
$$

$$
\begin{aligned}
t_3 &= -c_0{}^2 x a_1 - c_0 b_0 a_3 - a_0 y a_3{}^2 + c_0{}^2 a_2 - 2 c_0 a_1 x a_2 - c_0 a_1 y a_3 - \\
&\quad c_0 x a_2 y a_3 + c_0 a_0 x a_3 - c_0 a_0 y a_2 + c_0 b_0 y a_1 + y a_1 a_0 x a_2 + y^2 a_1 a_0 a_3 + \\
&\quad y a_1 b_0 a_2 + y^2 a_2 b_0 a_3 + x^2 a_3{}^2 c_0 - a_2{}^2 a_0 y - b_0 x a_3{}^2 y + c_0 a_3{}^2 + \\
&\quad c_0 a_2{}^2 + c_0 x^2 a_1{}^2 - x a_1{}^2 b_0 y.
\end{aligned}
$$

It is required to set all the above t_i to zero and solve. For the sake of clarity and efficiency, it was found to be convenient to attempt the calculation in stages. First Maple was asked to solve $t_1 = t_2 = 0$. Four solution sets were produced. I name these $\mathcal{T}$, $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$. In each case, all components of a, b, c not mentioned in the set are to be regarded as independent variables.

$\mathcal{T}$:

$$\{x = b_0/a_0, y = c_0/a_0\}.$$

This is the solution already referred to in above. For this case, the remaining constraints $t_0 = t_3 = 0$ are already true. The matrix A is

$$A = -2 \begin{bmatrix} a_1 + x a_2 + y a_3 & a_0 & x a_0 & y a_0 \\ -x a_3 + y a_2 - a_0 & a_1 & x a_1 & y a_1 \\ -y a_1 + a_3 - x a_0 & a_2 & x a_2 & y a_2 \\ -a_2 + x a_1 - y a_0 & a_3 & x a_3 & y a_3 \end{bmatrix}$$

whose eigenvalues take the form $(\lambda, \bar{\lambda}, 0, 0)$ for some complex λ. An eigenvalue set of this form indicates that the map is equivalent to a map in some complex subspace.

$\mathcal{U}$:

$$\begin{aligned} x &= \frac{a_2(a_1^2 + a_2^2 + a_3^2) - a_3(a_2 b_0 + a_0 a_1) + c_0(a_1^2 + a_2^2)}{a_1(a_1^2 + a_2^2 + a_3^2 t)}, \\ y &= \frac{a_3(a_1^2 + a_2^2 + a_3^2) + a_2(a_3 c_0 + a_0 a_1) - b_0(a_1^2 + a_3^2)}{a_1(a_1^2 + a_2^2 + a_3^2)}. \end{aligned}$$

This complicated case is a generalization of that found by Heidrich in [Heidrich, 1994]. Heidrich's example can be recovered from this solution by choosing $a_2 = a_3 = 0$ and renaming some of the remaining variables. Although rather complicated, Maple shows quite easily that for this solution set, the quantities t_0, t_3 are again automatically zero without further solving required. Calculation of the eigenvalues of A for this case once again reveals the structure of a complex-equivalent map.

$\mathcal{V}$:

$$x = b_0/a_0, \qquad y = \frac{a_2 a_0^2 + a_1(a_3 - b_0)a_0 + a_2 a_3 b_0}{a_0(a_2^2 + a_1^2)}.$$

This solution has $b = xa$ but c is not proportional to a. Unlike the previous sets, the quantities t_0, t_3 remain to be constrained to zero. I refrain from giving the expressions for these quantities after substitution of the solution so far, on account of their great length. However, Maple is still able to cope with solving the final two constraints and gives five possible solutions in terms of b_0, c_0 as follows:

1. $b_0 = a_2 a_0/a_1$, $c_0 = a_3 a_0/a_1$

2. $c_0 = (a_0{}^2 a_2 + a_3 a_0 a_1 - a_1 a_0 b_0 + a_2 a_3 b_0)/(a_2{}^2 + a_1{}^2)$

3. $c_0 =$

$$\frac{a_1(b_0 a_1^2 + b_0 a_3^2 + a_0^2 a_3) + a_2(-a_1^2 a_0 + b_0 a_2 a_1 - a_2^2 a_0 - a_3^2 a_0 + a_3 b_0 a_0)}{(a_2^2 + a_1^2) a_0}$$

4. $c_0 = a_0^2/a_2$, $b_0 = -a_1 a_0/a_2$

5. $c_0 = -(a_1{}^2 + a_2{}^2 + a_3{}^2)/a_2$, $b_0 = -a_1 a_0/a_2$.

All of these solution sets result in a complex eigenvalue pair and two zero eigenvalues, and so only yield complex-equivalent maps.

$\mathcal{W}$:

$$x = a_2/a_1, \; y = a_3/a_1.$$

Substitution of this solution set into the remaining expressions for t_0 and t_3 leads to just one remaining equation, since t_3 is found to disappear immediately. I find

$$a_1 t_0 =$$
$$2[b_0 a_2(a_1 a_0 + c_0 a_3) + a_1 c_0 a_3 a_0] - a_0^2(a_2^2 + a_3^2) - b_0^2(a_1^2 + a_3^2) - c_0^2(a_1^2 + a_2^2),$$

and on setting this to zero, we can choose to solve for any of the variables. I choose a_0 and find

$$a_0 = \frac{a_1(b_0 a_2 + c_0 a_3) \pm i(a_2 c_0 - a_3 b_0)\sqrt{a_1^2 + a_2^2 + a_3^2}}{a_2^2 + a_3^2}.$$

Here it has been assumed that a_2 and a_3 are not simultaneously zero and i has been used for the square root of -1, distinguishing it from the quaternion element $\mathbf{i}$. The case for which $a_2 = a_3 = 0$ may be solved as a special case. Furthermore, since we require all variables in the equations to be real, we now have to make the demand that $a_2 c_0 - a_3 b_0 = 0$. Yet again, after substituting all the necessary conditions into the matrix A and calculating the eigenvalues, I find only a complex-conjugate pair and a double zero. The special cases for which one or both of a_2, a_3 are set to zero also yield nothing new.

At this point, I have exhausted the solution set $\mathcal{M}$ and found only complex-equivalent maps. I must now turn to the set $\mathcal{N}$.

6.5.4 Solution set $\mathcal{N}$

The alternative solution set, $\mathcal{N}$, is much harder to analyse. In contrast to the previous solution set, $\det(A)$ is not automatically zero for this set; however, Maple quickly implies that the quantities t_1, t_2 and t_3 are already zero. This seems promising because there are only two conditions left, namely $t_0 = 0$ and $\det(A) = 0$. Unfortunately these conditions are very complicated and Maple does not give easily comprehensible output.

The required expressions can be regarded as respectively t_0, a third degree multinomial and $\det(A)$ a fourth degree multinomial in the nine variables a_0, b_0, c_0, a_1, b_1, c_1, b_2, c_2 and c_3. I choose to write both as quadratics in a_0:

$$
\begin{aligned}
t_0 \; = \; & -a_0^2(b_2 + c_3) + 2a_0(b_0 b_1 + c_0 c_1) - 3c_3(c_1^2 + c_2^2 - b_2^2) - \\
& b_2(c_0^2 + 3c_2^2 - 3c_3^2) - a_1(b_0^2 + 3b_1^2 - 3b_2^2 - 6b_2 c_3 + c_0^2 + 3c_1^2 - 3c_3^2) + \\
& 3a_1^2(b_2 + c_3) - b_0^2 c_3 + 2b_0 c_0 c_2 - 3b_1^2 b_2 - 6b_1 c_1 c_2
\end{aligned}
$$

and

$$
\begin{aligned}
\det(A) = & \\
& a_0^2(b_2 c_3 - c_2^2) + 2a_0(c_0(b_1 c_2 - b_2 c_1) - b_0(b_1 c_3 - c_1 c_2)) + a_1^2(b_2 c_3 - c_2^2) + \\
& a_1(b_0^2 c_3 - 2b_0 c_0 c_2 - b_1^2 c_3 + 2b_1 c_1 c_2 + b_2(b_2 c_3 + c_0^2 - c_1^2 - c_2^2 + c_3^2) - c_2^2 c_3) - \\
& b_0^2 c_1^2 - b_1^2(b_2 c_3 + c_0^2 + c_3^2) + 2b_0 b_1 c_0 c_1 + 2b_1 c_1 c_2(b_2 + c_3) - \\
& b_2 c_1^2(b_2 + c_3)
\end{aligned}
$$

where it is to be understood that the substitution $a_1 = (b_1^2 + c_1^2 + c_2^2 - b_2 c_3)/(b_2 + c_3)$ demanded by the regularity of the second iterate is to be made at a convenient moment.

Clearly one or other of these could be solved for say, a_0 and the expression for a_1 then substituted into the result; however, the remaining constraint then becomes very difficult to solve as it contains square roots of large expressions. Instead, I now return to our study of the symmetry properties of regular iterability, and invoke these to simplify our equations.

Recall that the most general symmetry which preserves regular iterability of a linear quaternion map, takes the form of a rotation of the pure part of q. One may write this as a rotation acting on A in the form MAM^{-1}, where M is a matrix representation for a general element of the group $SO(1,3)$. There are various ways in which M can be used to simplify our equations. In order to avoid having to

explicitly demonstrate that our choices are valid, I observe that the bottom right 3×3 block of M is just an $SO(3)$ rotation acting on the bottom right hand 3×3 block of A. Now since in the case $\mathcal{N}$, the bottom right 3×3 block of A is real symmetric, it can be diagonalized by appropriate choice of M, with the bottom right 3×3 block of M an orthogonal matrix. This operation is regularity-preserving. I may therefore assume that there are zeroes in the positions $A_{3,2}, A_{4,2}$ and $A_{4,3}$. Thus we may permit ourselves to set $b_1 = c_1 = c_2 = 0$ everywhere. The disappearance of these parameters in other positions within A is then guaranteed by the knowledge that the rotation preserves regular iterability. A now takes the form

$$
A = -2 \begin{bmatrix} -\frac{b_2 c_3}{b_2 + c_3} + b_2 + c_3 & a_0 & b_0 & c_0 \\ -a_0 & -\frac{b_2 c_3}{b_2 + c_3} & 0 & 0 \\ -b_0 & 0 & b_2 & 0 \\ -c_0 & 0 & 0 & c_3 \end{bmatrix}
$$

and the remaining two equations simplify to

$$
(b_2 + c_3)^2 a_0^2 - 3 b_2^2 c_3^2 + b_0^2 c_3^2 + c_0^2 b_2^2 = 0
$$

and

$$
b_2 c_3 \left[(b_2 + c_3)^2 a_0^2 - (b_2 + c_3)(b_2 c_0^2 + c_3 b_0^2) + b_2 c_3 (b_2^2 + b_2 c_3 + c_3^2) \right] = 0,
$$

where it has been assumed that $b_2 + c_3 \neq 0$, this case having been exhausted by earlier solutions.

Maple now gives twelve possible solution sets, but these can be collected into three groups of four, differing within each group only by the signs attached to square roots or in one case, by permutation of some of the parameters. The following three solution sets therefore represent all possible cases:

$$
\{b_2 = a_0 = b_0 = 0\},
$$
$$
\{c_0 \text{ and } a_0 = \text{complicated expressions}\},
$$
$$
\{a_0 = c_0 = 0, \ b_2 = -c_3/2, \ b_0 = \sqrt{3} c_3/2\}.
$$

Only one of these sets yields anything new. The first set gives a complex-equivalent map. The second requires a careful examination of the complicated expressions. It turns out that they cannot be real simultaneously. The set must

therefore be discarded. Substitution of the last solution set results in

$$A = -c_3 \begin{bmatrix} 3 & 0 & -\sqrt{3} & 0 \\ 0 & 2 & 0 & 0 \\ \sqrt{3} & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $\lambda(\lambda - c_3)^3$ so that the eigenvalues are just c_3 and zero.

The most general regularly iterable A of this new kind is now obtained by transforming the above A to MAM^{-1} with a general $SO(1,3)$ matrix M. The result is rather unwieldy so I refrain from writing the details. Suffice to say that such a transformation must leave the characteristic equation, and hence the eigenvalues, invariant.

Although this is the only example I have found of a non-complex-equivalent linear map, the dynamics are not interesting. A quick calculation shows that $A^2 = -c_3A$ so that the map projects any initial q into a particular 3-dimensional subspace of $\mathbb{H}$, then subsequent applications of f move the point in a straight line along the ray connecting this first iterate to the origin, converging to zero if $|c_3| < 1$ or diverging to infinity if $|c_3| > 1$. For $c_3 = 1$, f fixes q after the first iterate. This completes the analysis of regularly iterable linear quaternion maps.

6.6   Regularly iterable quadratic maps

Let us now see whether anything more interesting is possible with regular quadratic maps. Sudbery [Sudbery, 1979] shows that a basis for such functions is the set of polynomials

$$\begin{aligned}
P_{11}(q) &= [P_1(q)]^2 = (q_0\mathbf{i} - q_1)^2 \\
P_{22}(q) &= [P_2(q)]^2 = (q_0\mathbf{j} - q_2)^2 \\
P_{33}(q) &= [P_3(q)]^2 = (q_0\mathbf{k} - q_3)^2 \\
P_{12}(q) &= q_1q_2 - q_0(\mathbf{i}q_2 + \mathbf{j}q_1) \\
P_{13}(q) &= q_1q_3 - q_0(\mathbf{i}q_3 + \mathbf{k}q_1) \\
P_{23}(q) &= q_2q_3 - q_0(\mathbf{j}q_3 + \mathbf{k}q_2).
\end{aligned}$$

Thus the most general left-regular map which is real-homogeneous of degree 2 is

$$f(q) = P_{11}(q)a + P_{22}(q)b + P_{33}(q)c + P_{12}(q)d + P_{13}(q)e + P_{23}(q)f,$$

where a, b, c, d, e, f are constant quaternions. Note that the basis functions P_{nn} involve a projection onto a two real-dimensional subspace. It follows that a function such as

$$a + P_1(q)b + P_{11}(q)c$$

where a, b, c are constant quaternions, is left-regularly iterable, since it splits into independent maps, each conjugate to complex maps $\alpha + \beta z + \gamma z^2$ ($\alpha, \beta, \gamma, z \in \mathbb{C}$), in each of the planes 1-$\mathbf{i}$ and $\mathbf{j}$-$\mathbf{k}$. Of course, the dynamics of all these maps can be completely understood with the known theory of complex analytic dynamics. This class of regularly iterable maps may be enlarged by conjugation with Möbius transformations $\phi(q) = (aq + b)(cq + d)^{-1}$, which, as shown by [Sudbery, 1979], preserves regularity. However, the inclusion of terms in P_{12}, P_{13}, P_{23}, or the mixing of terms such as P_{11} and P_{22} destroys regularity even in the second iterate. I conclude that the requirement of regular iterability is too strong to allow the existence of maps other than those conjugate to complex quadratic maps.

6.7   Conclusion

I conclude this chapter by noting that the special properties of iteration of complex analytic maps have no analog in the quaternions. In particular, no theorem is known giving a quaternion analog to the well-known theorem that for complex rational maps each stable cycle attracts at least one critical point [Milnor, 1990]. The powerful results of complex discrete dynamical systems theory depend on the scalar and commutative nature of the underlying field. Thus, I have found no evidence that discrete dynamical systems on the quaternions have any interesting behaviour different from the complex case.

Nevertheless, I have explicitly exhibited all regularly iterable linear maps on the space $\mathbb{H}$ of quaternions. Earlier speculation about the nature of such maps has been largely substantiated, in that almost all the maps found are equivalent to linear maps in a complex subspace of $\mathbb{H}$. One unexpected exceptional class of maps has been found, but the dynamics were seen to be uninteresting. While performing this study, I have been motivated by the prospect of looking for regularly iterable nonlinear maps on $\mathbb{H}$, since if such maps exist, they could be expected to have a linearized form which is also regularly iterable.

Chapter 7

Formal power series algorithms

Antipholus of Syracuse: *'Transform me then, and to your power I'll yield'*

The Comedy of Errors, act 3, scene 2

Many calculations in this thesis depend crucially on formal power series manipulations. I collect together in this chapter some basic ideas of this topic, and then describe some new computer algorithms for manipulating formal power series and solving functional equations. Several of my results depend on these new algorithms. They were computed with software making use of my own C++ formal power series package, available at [Briggs, 1996].

7.1 Formal power series

Formal power series are just sequences with elements in a fixed field K. The set of such formal power series is denoted $K[[x]]$. I write either $a = [a_0, a_1, \ldots]$ or $\sum_{i=0}^{\infty} a_i x^i$, where $a_i \in K$. The field K will be the rationals in the examples I present, but it just as well could be complex numbers, quaternions, invertible n by n matrices, etc. The second notation is a formal device, and does not imply that x has any value, or that there is any infinite sum. The first element a_0 of the sequence is called the constant term. $K[[x]]$ is an integral domain (a ring without zero divisors). The operations are termwise addition and Cauchy multiplication ($c = ab \iff c_j = \sum_{i=0}^{j} a_i b_{j-i}$). Units (that is, invertible elements) are formal power series with nonzero constant term. These are formal power series having multiplicative inverses. *Series composition*, denoted $a \circ b$, is the substitution of b

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for x in a. The compositional identity formal power series is $[0, 1] \equiv x$. For $a \circ b$ to be well-defined, we require $b_0 = 0$, that is, b must be a nonunit. Any nonunit b has a compositional inverse (or reversion), denoted $b^{<-1>}$. It of course satisfies $b^{<-1>} \circ b = b \circ b^{<-1>} = [0, 1]$. The standard references on formal power series are [Knuth, 1981], [Henrici, 1984] and (particularly for applications) [Wilf, 1990].

7.2   Iteration theory and functional equations

Iteration theory is concerned with the behaviour of sequences generated by repeatedly applying a given map f to an initial point. Let $x_{n+1} = f(x_n) \equiv f^{<n+1>}(x_0)$, $n = 0, 1, 2, \ldots$, with x_0 given. The sequence $< x_0, x_1, x_2, \ldots >$ is the orbit with seed x_0.

Around 1870 Schröder [Schröder, 1871] proposed studying the orbit of x_0 under iteration of the map f, by trying to find a new coordinate system in which the orbit looks simpler. In particular, we can try to make the effect of f something that can be explicitly iterated, like the function 'multiply by a constant'. (The history of this idea can be read in Dan Alexander's book [Alexander, 1994].) This will be true if the map from old to new coordinates σ satisfies Schröder's functional equation for some constant s:

$$\sigma \circ f(x) - s\sigma(x) = 0 \qquad \forall x. \tag{7.1}$$

Note that this a functional equation; that is, an equation in which the unknown σ is a function. It follows that $f^{<k>}(x) = \sigma^{<-1>}(s^k \sigma(x))$ for all positive integers k. Thus the problem of iterating f is solved, if the function σ exists. The difficulty has been transferred to the problem of finding σ.

Schröder found several explicit solutions to his equation in terms of elementary functions. However, there are not many such cases. Here are some examples:

1. $f_1(x) = 2(x + x^2)$, $\sigma_1(x) = \log(1 + 2x)/2$

2. $f_2(x) = -2(x + x^2)$, $\sigma_2(x) = \sqrt{3}/2(\cos^{<-1>}(-1/2 - x) - 2\pi/3)$

3. $f_3(x) = 4(x + x^2)$, $\sigma_3(x) = (\sinh^{<-1>}(\sqrt{x}))^2$.

Actually, cases two and three are conjugate with respect to the function $h(x) = -3/2 - 2x$. By this I mean that $h \circ f_3 = f_2 \circ h$. It follows that $h \circ f_3^{<k>} = f_2^{<k>} \circ h$

for $k = 1, 2, 3, \ldots$, so that the two maps are completely equivalent from the point of view of iteration.

In the general case we try to find series solutions to the Schröder equation, the point being that if we can show that it has a unique (real or complex) analytic solution, then the series solution must represent this function. On the other hand, the series solution may not converge. It is then just a formal power series solution. A slightly different point of view is that we want to distort or perturb the complex plane to make the orbit look simpler. The Schröder series is the 'perturbation expansion' which does this. As more terms are generated, we get a more and more accurate representation of the change of coordinates. Convergence of these series is often a difficult question and will not be considered here. In this section I will discuss algorithms for finding formal power series solutions. Before doing so, I summarize several classical functional equations:

S Schröder equation [Schröder, 1871]: $\quad \sigma \circ f - s\sigma = 0$

I Inhomogeneous Schröder equation: $\quad \sigma \circ f - s\sigma - g = 0$

A Abel equation [Abel, 1964]: $\quad \alpha \circ f - \alpha - a = 0$

P Poincaré equation [Poincaré, 1890]: $\quad \pi \circ (p\times) - f \circ \pi = 0$

B Böttcher equation [Böttcher, 1904]: $\quad \beta \circ f - \beta^2 = 0$

J Julia equation [Julia, 1918]: $\quad \iota \circ f - f'\iota = 0.$

Here a, s and p are constants, and f is a given formal power series (as is g in case I), which may or may not be the Taylor expansion of some analytic function. $p\times$ represents the function 'multiply by p'. f must be a nonunit, which means that if it does represent an analytic function, then there is a fixed point at zero. The formal power series with a Greek name is the unknown. Of these, the Poincaré equation needs no discussion since it is converted to the Schröder functional equation by $\sigma = \pi^{<-1>}$. The Abel equation tries to make f look like the function 'add a' in the new coordinate system. The constant a may be taken to be unity. The Böttcher[1] equation tries to make f look like the squaring function. It has formal power series solutions iff $f = [0, 0, f_2, \ldots]$ with $f_2 \neq 0$. In the Böttcher case, one may easily generalize to $\beta \circ f - \beta^m = 0$, $\mathbb{Z} \ni m > 2$. The Julia equation is in some sense more

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[1]Böttcher is also known in French as Bötkher, and in Russian as Бётхеръ.

general than the others. f' is the formal (termwise) derivative of f. Also worth considering is the functional equation

$$v \circ f \;=\; f'v / \operatorname{ord}(f)$$
$$\operatorname{ord}(v) \;=\; \operatorname{ord}(f - x)$$

($\operatorname{ord}(f)$ being the degree of the first term of f having non-zero coefficient) which, according to [Brent and Traub, 1980], was proposed by Knuth. It also generalizes several of the classical equations.

Some conversions are available between these functional equations in the case when σ exists. The proofs are simple exercises.

J $\leftrightarrow$ S: $\iota = c\sigma/\sigma'$.

A $\leftrightarrow$ S: $\alpha = a \log_s \circ \sigma$.

B $\leftrightarrow$ I: $\beta(x) = x \exp(\sigma(x) - 1)$, $s = 2$, $g(x) = 1 + \log[f(x)/(x^2 f'(0))]$.

The books of Kuczma [Kuczma, 1968; Kuczma, Choczewski and Ger, 1990] are good references for the general theory of functional equations.

7.3   An algorithm for the Schröder equation

Let the given formal power series be $f = [0, f_1, f_2, \ldots]$. Of course f may be a polynomial of finite degree, but this makes no difference to the following discussion. We are trying to find a formal power series $\sigma = [0, \sigma_1, \sigma_2, \ldots]$. We plug this into the left-hand side of S, getting

$$[0, \sigma_1 f_1 - s\sigma_1, \sigma_1 f_2 - s\sigma_2 + \sigma_2 f_1^2, \ldots]$$

which should be zero. Thus $s = f_1$, σ_1 is arbitrary ($= 1$, say), $\sigma_2 = f_2/(f_1 - f_1^2)$, etc. How can we get a iterative formula from this? The key observation is that each unknown σ_k first appears in the coefficient of x^k, multiplied by $f_1 - f_1^k$. (Proof by induction.) Thus we see straight away that there is no solution if f_1 is zero or a root of unity. In particular, in the complex case we cannot have $f_1 = \exp(2\pi i \zeta)$ if ζ is rational.

I thus get a practical algorithm as follows. At the ith step, I compute the coefficient of x^i in $\sigma \circ f - s\sigma$, but with σ_i set to zero. This coefficient is then $\sigma_i(f_1 - f_1^i)$. I give the algorithm in Maple, where `coeff(f,x,j)` means the coefficient of x^j in the formal power series f. The inhomogeneous case is a trivial variant.

```
schroeder:=proc(f:polynom(anything,x),n:posint)
  local sigma,i,u,f1,fi:
  f1:=coeff(collect(f,x),x,1):
  sigma[1]:=1: u:=0: fi:=f:
  for i from 2 to n do
    u:=series(u+sigma[i-1]*fi,x,n+1):
    fi:=series(f*fi,x,n+1):
    sigma[i]:=coeff(u,x,i)/(f1-coeff(fi,x,i))
  od:
  convert([seq(sigma[i]*x^i,i=1..n)],'+')
end:
```

This is an $\mathcal{O}(n^3)$ algorithm, as there are three nested loops. Note that asymptotically faster algorithms are known (for example, see [Brent and Traub, 1980]), but these are very complex and not optimal for short series. Here is an example output: the Schröder series of $2x + 2x^2$ is

$$x - x^2 + 4/3x^3 - 2x^4 + 16/5x^5 - 16/3x^6 + 64/7x^7 - 16x^8 + 256/9x^9 + \cdots,$$

which is indeed the expansion of $\log(1 + 2x)/2$ as claimed in example 1 above.

7.4   An algorithm for the Böttcher equation

```
boettcher:=proc(f:polynom(anything,x),n:posint)
  local beta,B,i,u,f2,fi:
  f2:=coeff(collect(f,x),x,2):
  beta[1]:=f2: u:=0: B:=f2*x: fi:=f:
  for i from 2 to n do
    u:=series(u+beta[i-1]*fi,x,n+2):
    fi:=series(f*fi,x,n+2):
    beta[i]:=(coeff(u,x,i+1)-coeff(expand(B^2),x,i+1))/2/f2:
    B:=B+beta[i]*x^i
  od:
  B
end:
```

This algorithm is a variation on the same idea. Here β_i first appears in the coefficient of x^{i+1} in $\beta \circ f - \beta^2 = 0$, multiplied by $2\beta_1$. Setting $\beta_1 = f_2$ starts the iteration.

7.5   An algorithm for the Julia equation

Series of the form $x + f_2 x^2 + \mathcal{O}(x^3)$, $f_2 \neq 0$ have Julia series. More generally, we may consider $x + f_d x^m + \mathcal{O}(x^{m+1})$, $f_m \neq 0$, $\mathbb{Z} \ni m > 2$. Actually, the correspondence $f \leftrightarrow \iota$ is one-to-one, and so is invertible. The Julia series begins

$$[0, 0, 1, f_3/f_2 - f_2, \frac{3}{2}f_2^2 + f_4/f_2 - \frac{5}{2}f_3, \ldots].$$

The following algorithm computes any number of terms. Here ι_i first appears in the coefficient of x^{i+1} in $\iota \circ f - f'\iota = 0$ ($i \geq 3$), multiplied by $(i - 2)f_2$. Setting $\iota_2 = 1$ starts the iteration. In the following Maple code j represents ι.

```
julia:=proc(f:polynom(anything,x),n:posint)
  local df,iota,j,i,u,f2,fi:
  f2:=coeff(collect(f,x),x,2):
  df:=diff(convert(f,polynom),x):
  iota[1]:=0: iota[2]:=1:
  fi:=f*f: j:=x^2: u:=0:
  for i from 3 to n do
    u:=series(u+iota[i-1]*fi,x,n+2):
    fi:=series(f*fi,x,n+2):
    iota[i]:=(coeff(expand(df*j),x,i+1)-coeff(u,x,i+1))/(i-2)/f2:
    j:=j+iota[i]*x^i
  od:
  j
end:
```

As pointed out in Kuczma [Kuczma et al., 1990], Baker has proved that the Julia series rarely converges. In fact if it does for a meromorphic function f, then f must be of the form $x/(1 + bx)$ for some constant b. A simple example is $f = x/(1 - x)$, $\iota = x^2$.

7.6 The Chebyshev equation

The Chebyshev polynomials $T_n(x) \equiv \cos(n \arccos(x))$ are explicitly iterable since $T_n \circ T_m = T_{mn}$, so we might consider the 'Chebyshev functional equation' $\tau \circ f - T_n \circ \tau = 0$. There are no formal power series solutions, as is easily checked by trial substitution. However, it is easy to show that if $\psi(x) \equiv (\rho x + (\rho x)^{-1})/2$ where $\rho^{n-1} = 1$, then $\psi(x^n) = T_n \circ \psi(x)$. It follows easily that if we just solve the Böttcher equation of f for β, then $\tau = \psi \circ \beta$ solves the Chebyshev functional equation.

7.7 Iterative square roots

Here is a 'real' application of these algorithms. The problem is to compute an iterative square root of the function $f(z) = z^2 + 1$; that is, a function $f^{<1/2>}$ having the property that $f^{<1/2>} \circ f^{<1/2>} = f$. (Note that there is nothing special about this function; we could consider $z^2 + c$ for arbitrary complex constants c). Recall the following concepts: a fixed point is a point x^* such that $x^* = f(x^*)$; an n-cycle is an orbit which returns to its seed after n iterations, so that $x_0 = f^{<n>}(x_0)$.

First note that f cannot have a global iterative root, that is, one defined on the whole complex plane. We see this by noting that f has two distinct fixed points, and since $f^{<2>}$ is a quartic polynomial, f has precisely one (prime) two-cycle, $\{z_{\pm}^{**}\}$ say. Now suppose that $f^{<1/2>}$ is a global iterative root of f. Then

$$f^{<2>} \circ f^{<1/2>}(z_+^{**}) = f^{<1/2>} \circ f^{<2>}(z_+^{**}) = f^{<1/2>}(z_+^{**}).$$

Thus $f^{<1/2>}(z_+^{**})$ belongs to another two-cycle. But this is impossible as I have shown above that there is only one such. Thus any iterative root of f can only be defined on some subset of the complex plane, at least excluding $\{z_{\pm}^{**}\}$.

However, a formal power series solution to the Böttcher equation (which does in fact converge in a neighbourhood of infinity) may be constructed for the related function $F(x) \equiv 1/f(1/x)$. This is $\beta = x - x^3/2 + x^5/8 + 5x^7/16 - 101x^9/128 + 321x^{11}/256 + \cdots$. Now an iterative square root of F is given by $\beta^{<-1>}(\beta(x)^{\sqrt{2}})$ (this is easily checked), and of f by $\beta^{<-1>}(\beta(1/x)^{\sqrt{2}})$. Note that this function is not real analytic, and its domain cannot include the fixed points of f mentioned above.

Here is another example, using Julia's equation: having found ι, solve the equation $\iota \circ g - g'\iota = 0$ for g with the form $[0, 1, f_2/2, \ldots]$. Then $g \circ g = f$, that

is, g is an (at least formal) iterative square root of f.

For example, the function $f(x) = e^x - 1$ is known to have a iterative square root g real analytic on $(0, \infty)$. The Julia series ι of f is

$$[0, 0, 1, -1/6, 1/24, -1/90, 11/4320, -1/3360, \ldots],$$

which cannot converge for any $x > 0$ by Baker's result, and g is thus

$$[0, 1, 1/4, 1/48, 0, 1/3840, -7/92160, 1/645120, \ldots].$$

All the above algorithms can be rewritten so as not to use the Maple 'series' command, but with a considerable loss in efficiency. However, for conversion to other languages such as C, this may be useful. As an example, here is the Schröder algorithm with the series implemented as arrays:

```
# Auxiliary arrays... t=f^i, u=s@f
sigma:=array[1..n]: sigma[1]:=1:
t:=array[1..n]: u:=array[1..n]:
for i from 1 to n do t[i]:=coeff(f,x,i) od:
for i from 1 to n do u[i]:=sigma[1]*t[i] od:
f1:=coeff(f,x,1): f1i:=f1^2:
for i from 2 to n do # get sigma[i]
  for k from n to 1 by -1 do
    t[k]:=0:
    for j from 1 to k-1 do t[k]:=t[k]+coeff(f,x,j)*t[k-j] od
  od:
  sigma[i]:=u[i]/(f1-f1i):
  f1i:=f1i*f1:
  for k from 1 to n do u[k]:=u[k]+sigma[i]*t[k] od:
  lprint(sigma[i])
od:
```

7.8 A new algorithm for solution of functional equations using algorithmic differentiation

I describe in this section an application of algorithmic differentiation techniques (see Appendix B) to the solution in formal power series of linear functional equations. This gives a new algorithm which is not as fast as formal Newton-based

techniques, but is easily implemented and has a complexity $\max(\text{comp}(n), n^2)$, where $\text{comp}(n)$ is the complexity of computing n terms of the composition of two series.

Here I describe an easily-implemented algorithm which solves the general linear nth order functional equation for the unknown series ϕ:

$$\sum_{k=0}^{n} a_k(x) \phi \circ f^{<k>}(x) + b(x) = 0,$$

where a_k, b and f are given formal power series and, as usual, $f^{<k>}$ indicates the k-fold composition of f with itself. For composition to be well-defined, we require the constant term of f to be zero. This equation includes the standard cases of:

1. Schröder's equation ($n = 1, a_1 \equiv 1, a_0(x) = -f_1, b \equiv 0$);

2. Abel's equation ($n = 1, a_1 \equiv 1, a_0 \equiv 1, b \equiv -1$);

3. Julia's equation ($n = 1, a_1 \equiv 1, a_0(x) = -f'(x), b \equiv 0$);

4. Series reversion ($n = 1, a_1 \equiv 1, a_0 \equiv 0, b(x) = -x$); and

5. q-difference equations ($f(x) = qx$).

7.8.1   Method

The key observation is that the equations to be solved form a lower-triangular linear system. For example, in the case of Schröder's equation, we have

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ f_2 & f_1^2 - f_1 & 0 & \cdots \\ f_3 & 3f_1 f_2 & f_1^3 - f_1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

The lower triangularity follows from the fact that σ_k depends only on f_i for $i \leqslant k$. We see that given the matrix above, the desired solution is easily obtained by forward-substitution. The problem remains of how to compute the matrix. Though it is not immediately obvious, algorithmic differentiation techniques (see Appendix B) are ideal for this purpose. These methods rely on extending the computer representation of numbers to include the gradients of computed quantities with respect to specified independent variables. In our application, we notice that the gradients of the terms of the expression $\sigma \circ f(x) - f_1 \sigma(x)$ with respect

to the variables σ_i are precisely the desired matrix elements, and what is more, the dependence is linear, so that the gradients may be computed even though the values of the independent variables are unknown! Thus, the key step is the programming of standard algorithms for composition, multiplication, and addition of formal power series described in § 7.2 in the extended gradient arithmetic. I now give an example.

7.8.2   Example

Consider first Schröder's equation (7.1). To make the algorithm explicit, I exhibit the C++ code:

```
ps solve_schroeder(ps f){    // Returns Schroeder series of PS f
  int i, j, n = degree(f);   // n is number of available terms
  ps sigma(n);               // Construct sigma, initially zero
  sigma[0] = 0;              // Schroeder series, constant term
  sigma[1] = 1;              // arbitrary choice of scaling
  dim = n;                   // dim is number of independent variables
  for (i=1; i<=n; i++){      // declare sigma_i to be indpt var number i
    ad x(sigma[i],i);
    sigma[i] = x;       }
  ps d = sigma&f - f[1]*sigma; // & is power series composition
  for (i=2; i<=n; i++){        // Forward substitute
    double sum = 0;            // d[i][j] is Jacobian matrix element
    for (j=1; j<i; j++) sum += d[i][j]*sigma[j];
    sigma[i] = -sum/d[i][i]; }
  return sigma;
}
```

This code makes use of my own formal power series and automatic differentiation C++ software (see Appendix B), which defines types ps (power series) and ad (automatically differentiated) respectively. As a test, I computed the Schröder series for the three functions f given in § 7.2 for which the series converges and represents a known elementary function. In all cases the numerical solution agrees with the Taylor expansion of the exact solution. As another example, in which there is no exact solution in terms of elementary functions, I consider the computation of the iterative square root of $2(e^x-1)$, that is, a series ϕ such that $\phi\circ\phi(x) = 2(e^x-1)$.

Such a series is clearly given by $\sigma^{<-1>}(\sqrt{2}\sigma(x))$, where σ is the Schröder series of $2(e^x - 1)$. My algorithm computes

$$\phi(x) = 1.41421x + 0.292893x^2 + 0.02138x^3 + 0.00054x^4 + 0.000033x^5 + \mathcal{O}(x^7).$$

This series appears to converge for $|x| < 2$, though of course that fact cannot be established by my method.

7.9 More on Julia's functional equation

<div align="right">

Proteus: *'Why, this is the ring I gave to Julia'*

The Two Gentlemen of Verona, Act 5, scene 4

</div>

Julia's equation is $\iota \circ f - f'\iota = 0$, where f is a given element of the ring of formal power series over $\mathbb{C}$. $f_0 = 0$ is assumed throughout this section. Julia's equation is sometimes known as Jabotinsky's equation [Reich, 1985]. The Julia series ι is sometimes known as the associated series to f [Muckenhoupt, 1962], iterative logarithm [Kuczma et al., 1990], or generator [Labelle, 1980]. Its importance lies in applications to the study of the iteration of f. In particular, it can be used in place of Abel's equation to avoid the singularity at the origin that the Abel function of f possesses. I write $\iota = \text{julia}(f)$. If ι is a Julia series for f, then so is $c\iota$ for any nonzero constant c. Apart from this scale ambiguity, the correspondence $f \leftrightarrow \iota$ is one-to-one, and so is invertible. I will choose the normalization $\iota_m = f_m$, where $m = \text{ord}(f - x)$ is the index of the first nonzero coefficient of f.

Here are a few simple examples: If $f = x + f_2 x^2 + \mathcal{O}(x^3)$, $f_2 \neq 0$, then

$$\iota = (f_3 - f_2^2)x^2 + \left(\frac{3}{2}f_2^3 + f_4 - \frac{5}{2}f_3 f_2\right)x^3 + \cdots. \tag{7.2}$$

If $f = x/(1 - bx), b = \text{constant}$, then $\iota = bx^2$; and if $\iota = x^3 + x^5$, then $f = x + x^3 + 5/2x^5 + \cdots$.

7.9.1 Algorithms

Labelle [Labelle, 1980] gives the following formula:

$$\text{julia}(f) = \sum_{k \geq 1} (-1)^{k-1} \frac{1}{k} g_k(x),$$

where $g_0 = x, g_{k+1}(x) = g_k \circ f - g_k, k = 0, 1, \ldots$. (The similarity to the ordinary logarithm function is clear.) However, this produces an $\mathcal{O}(n^4)$ algorithm. I therefore designed the following $\mathcal{O}(n^3)$ algorithm:

```
// Solve Julia eqn for f.  (from ps.h).
// where f=x+fm*x^m+...+fn*x^n, fm!=0, 2<=m<=n.
template<class Scalar> ps<Scalar> julia(ps<Scalar>& f){
  int i,k,l,m=2,n=degree(f);
  while (!(fabs(f[m])>0.0) && m<n) m++;
  Scalar q,fm=f[m],one=1.0;
  if( n<m || fabs(f[0])>0.0 || fabs(f[1]-one)>0.0 ){
    cerr<<"Bad input to julia, exiting.\n"; exit(1); }
  ps<Scalar> j(n),u(n),fk(n); j[m]=one;
  fk=powint(shif(f,-1),m-1);
  for (k=m+1; k<=n; k++) { // get j[k]
    // terms 0..k-1 of j are now correct. fk is now (f/x)^(k-2)
    for (l=n-1; l>0; l--) { // fk <- (f/x)*fk;
      q=0.0; for (i=0; i<=l; i++) q=q+f[i+1]*fk[l-i]; fk[l]=q; }
    // fk is now (f/x)^(k-1), all terms correct
    // update j&f, storing result shifted left by m-1...
    for (i=k-1; i<=n+m-1; i++) u[i-m+1]=u[i-m+1]+(j[k-1]*fk[i-k+1]);
    // u is now (j&f)/x^(m-1).  Next compute (f'*j)[k+m-1]...
    q=0.0; for (i=m; i<k; i++) q=q+(i+1)*f[i+1]*j[k+m-1-i];
    j[k]=(q-u[k])/((k-m)*fm);
  } // got j[k]
  return j;
}
```

7.9.2   The Julia function and Lambert's W function

Theorem 8.5.3 in [Kuczma et al., 1990] tells us that if f is meromorphic and regular at the origin, and $f = x + f_2 x^2 + \ldots$, and julia(f) has positive radius of convergence, then $f(x) = x/(1 - bx)$. However, if we allow a branch point at the origin, then $f = x/(1 - arx^r)^{-1/r}$ has the convergent Julia series ax^r for $r = 2, 3, 4, \ldots$. In most

cases the Julia series diverges. It is, however, Borel summable in a suitable domain [Ecalle, 1981].

What about when the Julia series is a polynomial? Presumably then the series f diverges. However, we may compute the function f by solving the differential equation $f'(x) = \iota \circ f(x)/\iota(x)$. While thinking about these ideas, I discovered an interesting connection between Julia's functional equation and Lambert's W function [Corless, Gonnet, Hare, Jeffrey and Knuth, 1995]. I will briefly describe this here. Lambert's W function is the inverse of the function $t \to t \exp(t)$. I deal with formal power series of the form

$$f(x) = x(1 + b_k x^k + \mathcal{O}(x^{k+1})); \qquad b_k \neq 0, \qquad k > 0. \tag{7.3}$$

It is well known [Kuczma et al., 1990] that under formal conjugacy, such series fall into conjugacy classes labelled by k and a complex parameter known as the iterative residuum, which can be calculated as the coefficient of x^{-1} in the formal Laurent series $1/\mathrm{julia}(f)$. For example, for $k = 2$, the iterative residuum is

$$1 - b_3/b_2^2,$$

and for $k = 3$, it is

$$3/2 - b_5/b_3^2 + b_4^2/b_3^3.$$

It also follows from the results of [Labelle, 1980] that every formal power series of the above form is formally conjugate to a series whose Julia series is

$$\omega_{\alpha,\beta} \equiv \alpha x(x^k + \beta x^{2k}) \tag{7.4}$$

for some constants α and β. We may in fact choose $\alpha = 1$ by the normalization discussed above. β is a function of the iterative residuum.

We may therefore turn the question of finding the Julia series ι of f around; rather than proceeding from f to ι, we may take ι to be of the form (7.4), and ask what formal power series (v, say) has that Julia series. We then know that there is a formal conjugacy from f to v. However, then v may be computed directly from Julia's functional equation in the form

$$v'(x) = \frac{v^2(1 - bv)}{x^2(1 - bx)}$$

with the initial value $v(0) = 0$. Since this differential equation has a singular point at the origin, we need to add the condition $v'(0) = 1$ to pick out the solution of interest.

The exact solution of this is explicitly

$$bv(x)^{-1} = \begin{cases} 1 + W[+\exp\{\log(-b + 1/x) + (1/x - c)/b - 1\}/b] & x < 1/b \\ 1 & x = 1/b \\ 1 + W[-\exp\{\log(+b - 1/x) + (1/x - c)/b - 1\}/b] & x > 1/b \end{cases}$$

where W is Lambert's function, and c is a constant. Although it is not immediately obvious, v is differentiable on the whole positive real axis.

This function v is therefore a kind of universal function for the iteration of series of the form (7.3). It is plotted (for various values of b, and normalized so that $v(1) = 1$) in Figure 7.1.

All iterative properties of f may be deduced from v. And, of course, when the formal power series involved actually converge and represent analytic functions, this result gives the iterative behaviour of the functions. I believe that more useful results could be derived from this point of view.
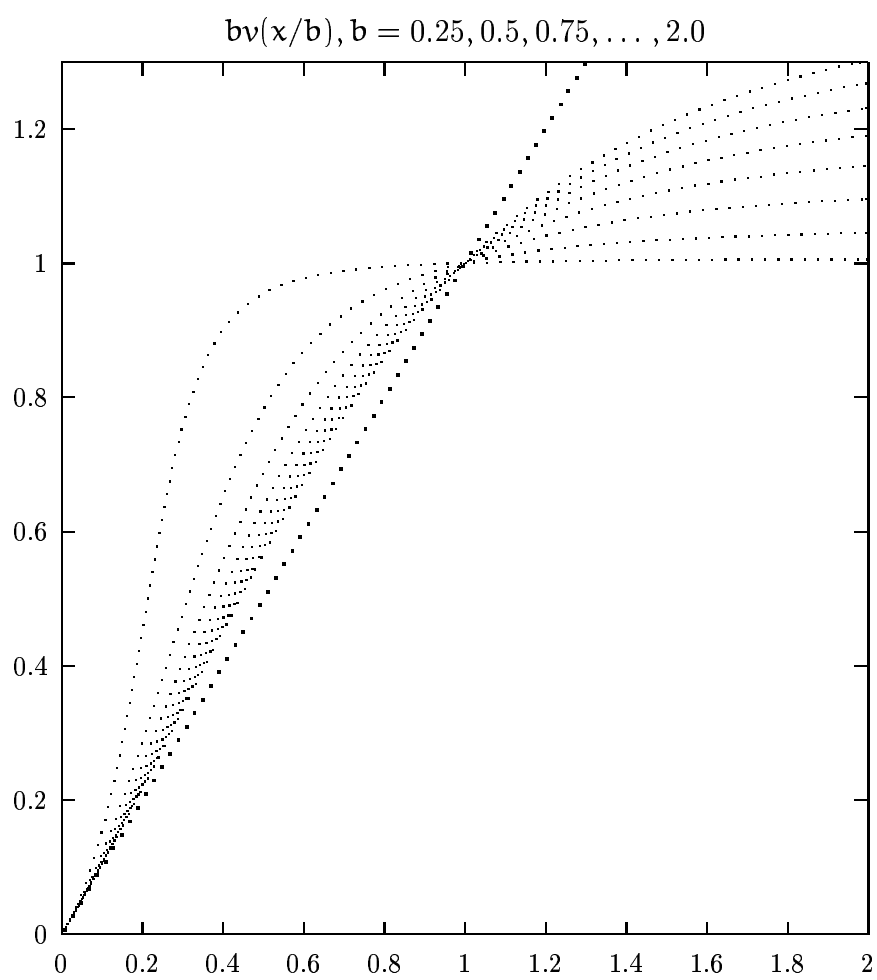
Figure 7.1: The universal Julia functions $\nu(x)$

Appendix A

How to calculate the Feigenbaum constants on your PC

(This appendix is included merely as a historical curiosity. It was the first article I ever wrote on the subject, and the one I still get most requests for! It is [Briggs, 1989])

A.1 Introduction

The family of functions $f_a : \mathbb{R} \to \mathbb{R}$ (dependent on the parameter $a \in \mathbb{R}$) defined by the map

$$x \rightarrowtail f_a(x) = a - x^2 \qquad (A.1)$$

has the property that there exist critical values a_i^* of a, at which bifurcations occur in the sets of limit points of sequences $\{x_i\}$ defined by the iteration

$$x_{i+1} = f_a(x_i), \qquad i = 0, 1, 2, \dots ; \qquad x_0 < \sqrt{a}. \qquad (A.2)$$

If the set of limit points for a given a has n elements, we describe the iteration as having an n-cycle. In other words, the sequence x_i is asymptotically periodic with period n. There exist cycles of each integer period [Keener, 1986]; amongst these we are especially interested in the superstable n-cycles, that is, those that contain 0 as one of the cycle points. Since the maximum of f occurs at 0, it follows that the stability

$$\Lambda_n(a) \equiv \prod_{i=0}^{n} \frac{df_a(x_i)}{dx} \qquad (A.3)$$

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is zero at a superstable n-cycle.

Let a_i^* be the least value of a at which a bifurcation to period 2^i occurs. It is known from the work of Feigenbaum [Feigenbaum, 1980d] that the sequence

$$\delta_i \equiv \frac{a_{i-1}^* - a_{i-2}^*}{a_i^* - a_{i-1}^*}, \qquad i = 2, 3, 4, \ldots \tag{A.4}$$

is convergent to a value $\delta \approx 4.669$. We describe in this note a direct method of calculation of δ. Previous methods have either used a numerical search for bifurcation values a_i^*, which is unreliable because the limited precision of computer arithmetic introduces artificial periods into the sequence x_i, or methods using power series approximations [Feigenbaum, 1980d]. The number δ is of interest in several physical and biological problems [Cvitanović, 1984; Briggs, 1987] which are modelled by equation (A.1). For example, it is equivalent to the logistic equation of population dynamics. Of course in practice a few digits of δ are sufficient. Nevertheless, the problem of calculating δ to many places has the same fascination as did the calculation of π to earlier generations of mathematicians.

A.2   The direct method

We consider the sequence of polynomials in a defined by

$$b_k(a) = a - [b_{k-1}(a)]^2, \qquad k = 1, 2, 3, \ldots \tag{A.5}$$

$$b_0(a) = 0. \tag{A.6}$$

The following property makes these polynomials useful for our purposes.

Lemma 3 *Let* $k = 2^n$. *Then* f_a *has a superstable* k-*cycle iff* $b_k(a) = 0$.

The proof is trivial. Thus superstable 2^n-cycles occur at zeros of b_{2^n}. We denote by a_i the least parameter value at which a superstable 2^i-cycle occurs. Clearly a bifurcation value a_i^* must occur between a_{i-1} and a_i. We conjecture that the stars can be removed in equation (A.4) without change to the limit δ, although we do not attempt to prove this. We will instead calculate as if δ were defined with superstable values a_i in equation (A.4), and see whether our δ agrees with that given by Feigenbaum.

Thus we can calculate δ to arbitrary precision by locating zeroes of the polynomials b_k. For this purpose Newton's method is satisfactory, so that the complete

method is:

$$a_i^0 = a_{i-1} + \frac{a_{i-1} - a_{i-2}}{\delta_{i-1}}, \qquad i = 2, 3, 4, \ldots \tag{A.7}$$

$$a_i^{j+1} = a_i^j - \frac{b_{2^i}(a_i^j)}{b'_{2^i}(a_i^j)}, \qquad j = 0, 1, 2, \ldots \tag{A.8}$$

$$b'_k(a) = 1 - 2b'_{k-1}(a)b_{k-1}(a), \qquad k = 1, 2, 3, \ldots \tag{A.9}$$

$$a_i = \lim_{j \to \infty} a_i^j \tag{A.10}$$

$$\delta_i = \frac{a_{i-1} - a_{i-2}}{a_i - a_{i-1}} \tag{A.11}$$

$$\delta = \lim_{i \to \infty} \delta_i \tag{A.12}$$

The first equation produces an initial approximation to the next superstable a value, which is refined by the Newton iteration; b'_k is the derivative of b_k. Thus a_i^j is a sequence convergent to the ith zero of b_{2^i}. The process was programmed in Turbo Pascal version 4.0, using extended precision, and started with $a_0^0 = 0; a_1^0 = 1; b'_0 = 0$ and $\delta_1 = 3.2$.

The rate of convergence of appears to be roughly linear, so that about one more significant decimal digit is gained every two iterations. The results were:

i	a_i	delta_i
2	1.3107026413	3.21851142203809
3	1.3815474844	4.38567759856834
4	1.3969453597	4.60094927653808
5	1.4002530812	4.65513049539198
6	1.4009619629	4.66611194782857
7	1.4011138049	4.66854858144684
8	1.4011463258	4.66906066064834
9	1.4011532908	4.66917155537963
10	1.4011547825	4.66919515602875
11	1.4011551020	4.66920022907521
12	1.4011551704	4.66920131316059
13	1.4011551851	4.66920154839814

The algorithm depends on finding the correct zero by Newton's method of a high degree polynomial with many closely spaced zeros. Thus it will fail if the initial approximation is not close enough to the required zero. This is the limiting factor determining the maximum precision of the above results.

A.3   Feigenbaum's α

If d_k denotes the value of the nearest cycle element to 0 in the superstable k-cycle, then the sequence

$$\alpha_i = \frac{d_i}{d_{i+1}}, \qquad i = 1, 2, 3, \ldots \tag{A.13}$$

is convergent to a value about 2.502. This constant is most easily calculated by realizing that the derivative b' defined above must satisfy

$$\lim_{i \to \infty} \frac{b'_{i+1}(a_{i+1})}{b'_i(a_i)} = \delta/\alpha. \tag{A.14}$$

(To see this, consider the slope of the line joining successive 'corners' of the graph of the figure.) Taking the calculation as far as b_{20} gave $\alpha = 2.502907875095$.

Appendix B

Some computational techniques

The work described in this thesis makes considerable use of a variety of novel computational techniques. This appendix briefly discusses some of them. Most of my computing was done in C++ [Stroustrup, 1991], except for the occasional use of fortran when I needed to use very high precision, for which I used Bailey's mpfun [Bailey, 1993b]. C++ has many advantages for this type of work, principally the ability to have user-defined types known as *classes* with overloaded operators and functions (that is, redefinition of symbols such as $+, -, *, /, exp, sin, \dots$). Also very useful is the *template* facility which allows parameterized types. I thus wrote classes for formal power series (Chapter 7), algorithmic differentiation, complex numbers, vectors, matrices etc. One of my most useful classes was quad, which implements 32 decimal place floating-point arithmetic. This allowed me to verify many results first computed in double (16 decimal place) arithmetic. In fact, some results involving iterated maps could not be obtained at all in double arithmetic. All of this software is available at http://www.pd.uwa.edu.au/Keith/.

Algorithmic (or automatic) differentiation methods rely on extending the computer representation of floating-point (real or complex) numbers to include the gradients of computed quantities with respect to specified independent variables. The chain rule is then used to propagate gradients through a series of computations. This is most naturally achieved in languages such as C++ which allow operator and function overloading. In this case, the computation of gradients becomes completely transparent to the programmer. For a recent survey of the field, see [Griewank and Corliss, 1991].

As an example of the usefulness of C++ techniques, here is the code used in § 2.4 and § 4.3 to explicitly form the Borel sum of an asymptotic series.

```
// Sum series 'a' by Loeffel's method.
// Returns estimate of a(z).
// Result should be independent of beta.
// K M Briggs 95 May 10

#include "ps.h" // defines class template ps

template<class Scalar, class Float>
Float Loeffel(const ps<Scalar>& a, const Float z, const Float beta){
  Float zero=0.0, one=1.0;
  if (z==zero) return a[0];
  if (z<zero)
    { cerr<<"Loeffel: z<0 not implemented.\n"; exit(1); }
  if (beta<=zero)
    { cerr<<"Loeffel: beta<=0, quitting.\n"; exit(1); }
  int i,k,n=degree(a);
  ps<Scalar> f(n), sigma(n), ft(n);
  Float q, fact=one;
  f[0]=a[0]; sigma[0]=zero;
  for (i=1; i<=n; i++)
    { q=i; fact=q*fact; f[i]=a[i]/fact; sigma[i]=beta/q; }
  ft=f&sigma; // & is series composition
  Float y=0.0,bz=1.0+beta/z;
  for (i=n; i>0; i--) { k=i+1; y=k*y/(i+bz)+ft[i]; }
  return y/bz+ft[0];
}
```

Appendix C

Curriculum Vitae

I was born in Cambridge, England and educated at the University of Adelaide, where I completed an Honours degree in Mathematical Physics. I have held research and teaching posts in physics and mathematics as follows:

84-85 Demonstrator in Physics, Coleg Prifysgol Gogledd Cymru.

86-88 Tutor in Physics, University of Adelaide.

89-90 Research Associate in Mathematics, La Trobe University and Royal Melbourne Institute of Technology.

91-92 Research Fellow in Mathematics, University of Melbourne.

93-94 Associate Lecturer in Applied Mathematics, University of Adelaide.

95-96 Research Associate, University of Western Australia.

97-   Postdoctoral Research Associate, University of Cambridge.

My research has been in in several areas: experimental chaos theory, relativistic electrodynamics, Feigenbaum universality in discrete dynamical systems, lattice models in statistical mechanics, colloid simulation, nonlinear time series analysis, quasilattice random walk theory, and musical acoustics.

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