The work of George Szekeres on functional equations

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Szekeres_seminar_QMUL_2006jan10.tex TYPESET 2006 JANUARY 12 13:41 IN PDFLATEX ON A LINUX SYSTEM

George (György) Szekeres 1911-2005

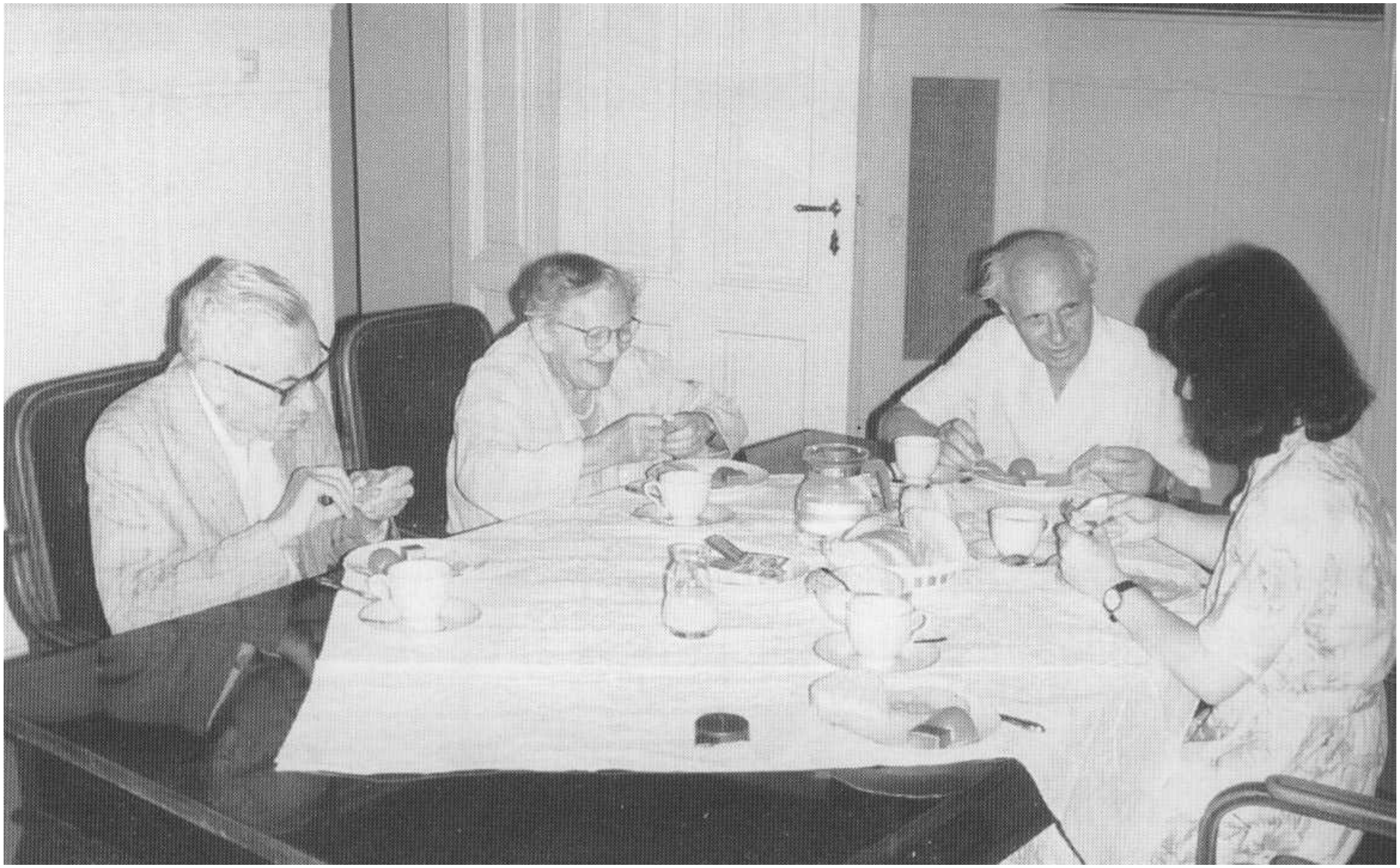
George Szekeres

- ★ studied chemical engineering at Technological University of Budapest
- ★ refugee in Shanghai 1940s
- ★ Adelaide 1948-63
- ★ UNSW 1963-75
- ★ died Adelaide 2005 Aug 28

The work of George Szekeres

- ★ first co-author of Erdős
- ★ graph theory
- ★ general relativity
- ★ functional equations
- ★ multi-dimensional continued fractions
- ★ lots more . . .

Szekeres and colleagues



Paul Erdős, Esther Klein, George Szekeres, Fan Chung

A letter from George

THE UNIVERSITY OF NEW SOUTH WALES

George Szekeres
Emeritus Professor
SCHOOL OF MATHEMATICS

13 Jan 97

Dear Keith,

I made some experiments myself and I think I see more clearly what is happening. First, the (totally real) field of $2\cos\frac{2\pi}{7}$: whatever pair I take, your recursion breaks down after a sufficient number of steps. Take for instance $\varphi_1 = -1 - 2\cos\frac{6\pi}{7}$, $\varphi_2 = 2\cos\frac{2\pi}{7} - 1$, and the unit $\beta = 4\theta^2 + 9\theta + 3 = 20.4$ ($\theta = 2\cos\frac{2\pi}{7}$) satisfying $\beta^3 = 20\beta^2 + 9\beta + 1$. (This is the critical unit). As long as the denominators are less than 10^{10} things are, as you have found, everything is O.K. It is rather unfortunate that you stopped there; I imagine the reason is that you multiplied φ_1 and φ_2 by $1, 2, \ldots 10^{10}$ successively and tested each one to see if they were best approximants. I have produced the best approximants by using my algorithm (2-dim. continued fraction); I could easily go up to 10^{50}, the computation itself took immeasurably small time, much less than a second. for a given pair I can give you the critical denominators, you can easily verify that they are best approximants.

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3 classes

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Outline

★ analytic iteration

★ Schröder & Koenigs

★ Szerekes on the Schröder and Abel equations

★ Szerekes on the Feigenbaum functional equation

★ Szerekes on Abel's equation and growth rates

★ formal iteration & Julia's equation (my speculations)

★ Jacobian conjecture (my speculations)

Iteration theory and functional equations

- map $f : \mathbb{X} \to \mathbb{X}$
- orbit $x_n = f(x_{n-1}) \equiv f^{<n>}(x_0)$, $n = 1, 2, 3, \ldots$
- around 1870 Schröder proposed studying the orbit by trying to find a new coordinate system in which the orbit 'looks simpler'
- simplest case: explicit iterability
- $\sigma \circ f(x) - f'(0)\sigma(x) = 0 \qquad \forall x$
- $\sigma \circ f^{<n>}(x) = (f'(0))^n \sigma(x), \quad n = 0, 1, 2, \ldots$

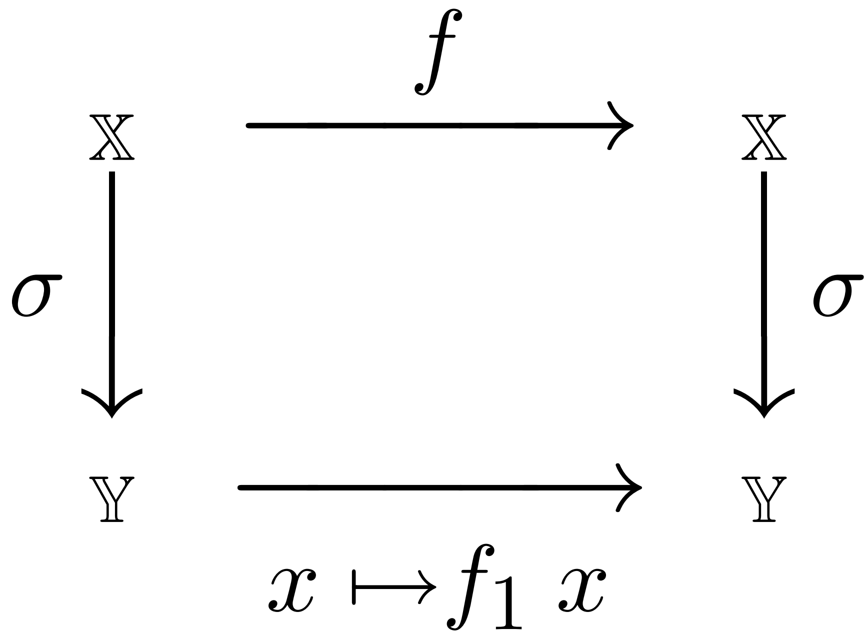
Friedrich Wilhelm Karl Ernst Schröder 1841-1902

- Pforzheim
- *Ueber iterirte Functionen* Math. Annalen **3** 296-322 (1871)

Solutions of the Schröder equation

- Schröder found several explicit solutions to his equation in terms of elementary functions
- $f_1(x) = \quad 2(x+x^2) \Rightarrow \sigma(x) = \log(1+2x)/2$
- $f_2(x) = -2(x+x^2) \Rightarrow \sigma(x) = \sqrt{3}/2(\arccos(-1/2-x)-2\pi/3)$
- $f_3(x) = \quad 4(x+x^2) \Rightarrow \sigma(x) = (\operatorname{arcsinh}(\sqrt{x}))^2$
- cases 2 and 3 are conjugate with respect to the function $h(x) = -3/2-2x$. That is, $h\circ f_3 = f_2\circ h$

The Schröder equation



- ★ $\sigma \circ f(x) = f_1 \sigma(x)$ **(Schröder, $f_1 \equiv f'(0)$)**
- ★ $\alpha \circ f(x) = \alpha(x) + 1$ **(Abel)**
- ★ $\beta \circ f(x) = (\beta(x))^2$ **(Boettcher=**Бётхеръ**)**
- ★ $\iota \circ f(x) = f'(x)\iota(x)$ **(Julia)**

Formal solution of the Schröder equation

- for $f(z)=\sum_{i=1}^{\infty} f_i\, z^i$ and $\sigma(z)=\sum_{i=1}^{\infty} \sigma_i\, z^i$ we have

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ f_2 & f_1^2-f_1 & 0 & \cdots \\ f_3 & 3f_1f_2 & f_1^3-f_1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

- defn: family of (continuous) iterates: $f^{<s>}(z)=\sigma^{<-1>}(f_1^s\sigma(z))$
- obtain formal solution from $f^{<s>}\circ f=f\circ f^{<s>}$

Schröder, Abel, Boettcher, and Julia

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\;f\;} & \mathbb{X} \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ \mathbb{X} & \xrightarrow[x \mapsto f_1 x]{} & \mathbb{X} \end{array} \qquad \begin{array}{ccc} \mathbb{X} & \xrightarrow{\;f\;} & \mathbb{X} \\ \alpha \big\downarrow & & \big\downarrow \alpha \\ \mathbb{X} & \xrightarrow[x \mapsto x+1]{} & \mathbb{X} \end{array}$$

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\;f\;} & \mathbb{X} \\ \beta \big\downarrow & & \big\downarrow \beta \\ \mathbb{X} & \xrightarrow[x \mapsto x^2]{} & \mathbb{X} \end{array} \qquad \begin{array}{ccc} \mathbb{X} & \xrightarrow{\;f\;} & \mathbb{X} \\ \iota \big\downarrow & & \big\downarrow \iota \\ \mathbb{X} & \xrightarrow[x \mapsto f'(x)\, x]{} & \mathbb{X} \end{array}$$

Gabriel Koenigs 1858-1931

- let $f(z)=\sum_{i=1}^{\infty} f_i\, z^i$ be analytic with $|f_1|<1$
- then the Schröder equation has an analytic solution $\sigma(z)=\sum_{i=1}^{\infty} \sigma_i\, z^i$, where each σ_k depends only on f_i for $i \leqslant k$
- also $\kappa(z) \equiv \lim_{i\to\infty} f_1^{-i} f^{<i>}(z)$ exists and satisfies $\kappa \circ f(z) = f_1 \kappa(z)$
- Kneser: sufficient to have $f(z)=f_1\, z+\mathcal{O}(|z|^{1+\delta})$, $\delta>0$ as $z\to 0$
- Szekeres [1]: for $f(z)=\frac{z}{2}+\frac{z^2}{3\pi}\sin\left(\frac{\pi}{|z|}\right)$, σ has 'flat spots'
- Szekeres: if $f:\mathbb{R}\to\mathbb{R}$ is continuous, strictly monotone increasing, $f'(x)$ exists and $f'(x)=a+\mathcal{O}(x^\delta)$, then the Koenigs function κ exists and is invertible

Solution of the Schröder equation [1]

★ cases for f:

- ▷ $f(z) = \sum_{i=m}^{\infty} f_i\, z^i$ *analytic*, $m > 0$
- ▷ $f(z) \sim \sum_{i=m}^{\infty} f_i\, z^i$ *as* $z \to 0$, $m > 0$
- ▷ f *a continuous real function*

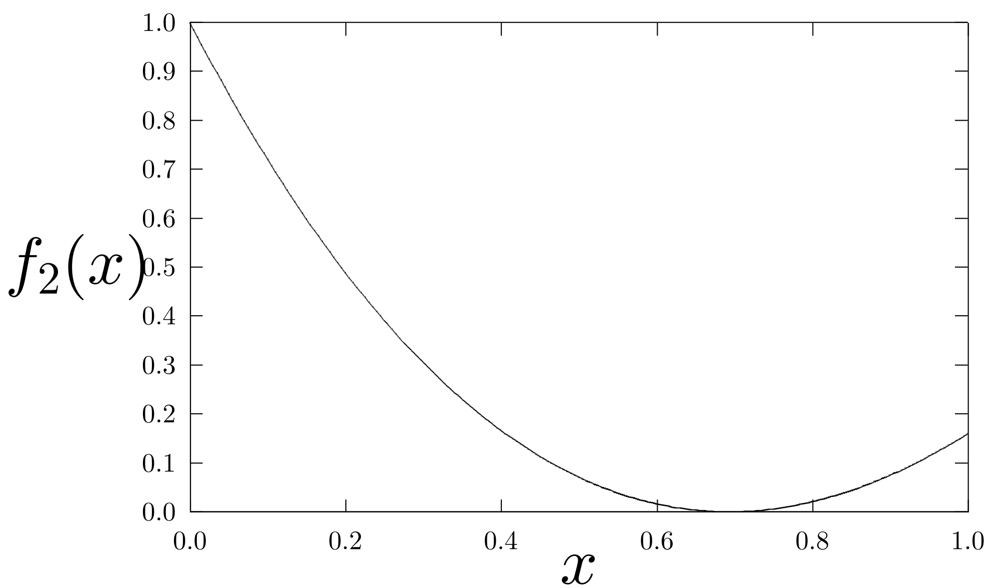
★ case $f_1 \neq 0$, $|f_1| \neq 1$: formal series exists and converges; continuous iterates are analytic at 0

★ case $|f_1| = 1$:

- ▷ f_1 *a root of unity: no formal solution*
- ▷ f_1 *not a root of unity: convergence depends on arithmetic conditions; e.g. Siegel* $\log|f_1^n - 1| = \mathcal{O}(\log(x))$ *as* $n \to \infty$ *sufficient*
- ▷ *Baker* $f(z) = \exp(z) - 1$: $f^{<s>}$ *exists formally but diverges unless* $n \in \mathbb{Z}$
- ▷ *Szekeres: there exists exactly one* $f^{<s>}$ *of which the formal series is an asymptotic expansion as* $z \to 0$
- ▷ *idea: use Abel equation as* $\sigma(z) = \exp(\alpha(z))$, $\alpha(z) \sim -1/z$
- ▷ *led to work by Écalle (Borel summability), Milnor (precise formulation of linearizability) and others*

The Feigenbaum functional equation 1

- consider scaling in the sequence of period-doubling bifurcations of maps $x \mapsto \mu - x^n$: we have orbit scaling α and parameter scaling δ
- solve for f: $f(x) = \gamma^{-1} f \circ f(\gamma x)$, $\gamma \equiv \alpha^{-1} = f(1)$
- $f(b) = 0$, $f(x) \sim c(b - x)^n$ for x near b
- there exist such regular solutions for each $n > 1$



The Feigenbaum functional equation 2

- ★ what about cases where $f \to 0$ faster than any power of x near b?
- ★ Szekeres' idea [8]: convert to coupled system, one of which is a Schröder or Abel equation:
  - ▷ *regular case:*

$$\begin{aligned} f \circ h(x) &= \gamma^2 f(x) \\ h(x) &= f(\gamma f(x)) \end{aligned}$$

  - ▷ *singular case: set* $A(x) \propto \log(f(b-x))$ *- we get* $A(x) = c_{-2}/x^2 + c_{-1}/x + c_0 \log x + c + c_1 x + c_2 x^2 + \cdots$
- ★ the singular series are divergent but Borel summable: we get $\gamma_\infty = 0.0333810598\ldots$ and $\delta_\infty = 29.576303\ldots$
- ★ Briggs & Dixon also solved the circle map case [8,9]
- ★ $g \circ g(\epsilon^2 x) = \epsilon g(x), \quad \epsilon = g(1)$, obtaining $\epsilon_\infty = -0.275026971\ldots$

Abel's equation and regular growth

- we work here with real functions on $[0, \infty)$
- Abel: $\alpha \circ f(x) = \alpha(x) + 1$
- Abel: if α and α_1 are strictly increasing C^1 solutions, then $\alpha(x) - \alpha_1(x) = \psi \circ \alpha(x)$, where ψ is 1-periodic
- Lévy: which is the 'best' solution?
- let $c \geqslant 1$, $\mathcal{C}_c$ be the set of strictly convex analytic functions with $f(0) = 0$, $f'(0) = c$, $f''(x) > 0$, $\mathcal{C} = \cup_c \mathcal{C}_c$
- principal Abel function (best behaviour at 0): for $f(x) = \sum_{i=1}^{\infty} f_i\, x^i \in \mathcal{C}_c$:
  - ▷ $\alpha(x) = \log_c(x) + \mathcal{O}(x)$ *if* $f_1 = c > 1$
  - ▷ $\alpha(x) = -\frac{1}{f_2 x} \log(x) + \mathcal{O}(x)$ *if* $f_1 = c = 1$
- we have thus selected a solution by its behaviour at 0. Szekeres wants to study the behavior at ∞

More functional equations

- ★ $D(x) \equiv 1/\alpha'(x)$ satisfies $D \circ f(x) = f'(x)D(x)$ (Julia)
- ★ $t(x) \equiv \sum_{k=0}^{\infty} 1/f^{<k>'}(x)$
- ★ t satisfies $t \circ f(x) = f'(x)(t(x)-1)$
- ★ if $f \in \mathcal{C}_c, c > 1$
  - ▷ $\alpha(x) = \log_c(x) + \mathcal{O}(x)$
  - ▷ $D(x) = \log(c)\left(x + \frac{f_2}{c(c-1)}x^2 + \dots\right)$
  - ▷ $E \circ f(x) = f'(x)(E(x) + D'(x))$
  - ▷ $\phi(x) = \frac{1}{D(x)}\left((t(x)-1)D'(x) - t'(x)D(x) + E(x)\right)$
- ★ if $f \in \mathcal{C}_1$
  - ▷ $\alpha(x) = -1/(f_2 x) + \mathcal{O}(\log(x))$
  - ▷ $D(x) = f_2 x^2 + (f_3 - f_2^2)x^3 + \dots$
  - ▷ $E \circ f(x) = f'(x)E(x) + D' \circ f(x)$
  - ▷ $\phi(x) = \frac{1}{D(x)}\left(t(x)D'(x) - t'(x)D(x) + E(x)\right)$

Yet more functional equations

- $\phi \circ f(x) = \phi(x) \quad \forall x \geqslant 0$
- $\psi(x) \equiv \phi(\alpha^{<-1>}(x))$ is 1-periodic, if α is the principal Abel function of f
- Szekeres' regularity criterion:
  - $\hat{\psi}_n \equiv \int_0^1 \exp(2\pi i n s)\, \psi(s) ds = \exp(\alpha_n + 2\pi i \beta_n)$
  - *this defines a mapping (LF sequence)* $(\mathcal{C}) \longrightarrow$ *real-valued sequences* α_n
  - *defn: a sequence is completely monotonic if all differences of all orders are non-negative*
  - *defn: a sequence is L-regular if its first differences are the difference of two completely monotonic bounded sequences*
  - *equivalent to being a moment sequence:* $\alpha_n = \int_0^1 t^n d\chi(t)$
  - *defn:* f *is regularly growing if its LF sequence is L-regular*

Szekeres' experimental results

- ★ $f(x)=2x+x^2$: L-regular
- ★ $f(x)=(1+x)^{e^2}-1$: probably L-regular
- ★ $f(x)=3x+x^2$: not L-regular
- ★ $f(x)=\exp(cx)-1$: probably L-regular
- ★ $f(x)=\exp(x)+x-1$: not L-regular
- ★ much more work needed to verify and extend these results!

Some speculation by KMB on Julia's equation

- ⋆ $\iota\circ f = f'(x)\iota(x)$ for formal series $f(z)=\sum_{i=1}^{\infty} f_i\, z^i$
- ⋆ if $f_2\neq 0$, then $\iota = (f_3-f_2^2)x^2+\left(\frac{3}{2}f_2^3+f_4-\frac{5}{2}f_3f_2\right)x^3+\cdots$
- ⋆ Julia inverse problem: any such series ι is formally conjugate to $\omega_{a,b}\equiv ax(x^k+bx^{2k})$ for appropriate a,b
- ⋆ we can thus solve the ODE (for $k=1$): $v'(x)=\frac{v^2(1-bv)}{x^2(1-bx)}$
- ⋆ The exact solution of this is explicitly

$$bv(x)^{-1} = \begin{cases} 1+W[+\exp\{\log(-b+\frac{1}{x})+(\frac{1}{x}-c)/b-1\}/b] & x<\frac{1}{b} \\ 1 & x=\frac{1}{b} \\ 1+W[-\exp\{\log(+b-\frac{1}{x})+(\frac{1}{x}-c)/b-1\}/b] & x>\frac{1}{b} \end{cases}$$

where W is Lambert's W function ($W(z)\exp(W(z))=z$)

More speculation by KMB on Julia's equation

- Jacobian conjecture: a polynomial map $f:\mathbb{R}^n\to\mathbb{R}^n$ with Jacobian determinant $\det J(f)$ equal to unity has a polynomial inverse
- we can write formally $f(z)=\exp(\omega D)z$, where D is the gradient operator and ω satisfies $J(f)\omega=\omega\circ f$ (multi-dimensional Julia equation)
- the mapping $f\mapsto\omega$ is a bijection (Labelle)
- then $f(z)=\exp(-\omega D)z$ ($z\in\mathbb{R}^n$)
- $\det J(f)$ can be expressed in terms of ω only
- we thus have a reformulation of the Jacobian conjecture: show that for all appropriate ω, both $\exp(\omega D)z$ and $\exp(-\omega D)z$ are polynomial

References

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