Integrals involving erf

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Abstract

I compute some integrals involving erf and apply them to some problems related to the maximum of Gaussian RVs.

1 Introduction

Let

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-(x-\mu)^2/2\sigma^2}$$

be the $\mathrm{N}(\mu,\sigma^2)$ Gaussian density, and

$$F_{\mu,\sigma^2}(x) = \frac{1}{2}\left[1+\mathrm{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

be its cumulative distribution. Here

$$\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

2 Known integrals

Most of these are not useful but are given for reference. Some obvious conditions on α, β etc. are not stated. The first integral will be used in section 3.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(\alpha x+\beta)^2}\,\mathrm{erf}\,(\gamma x+\delta)\,dx &= \frac{\sqrt{\pi}}{\alpha}\,\mathrm{erf}\left[\frac{\alpha\delta-\beta\gamma}{\sqrt{\alpha^2+\gamma^2}}\right] \\
\int_0^{\infty} e^{-\alpha^2x^2}\,\mathrm{erf}\,(\beta x)\,dx &= \frac{\arctan\,(\beta/\alpha)}{\alpha\sqrt{\pi}} \\
\int_0^{\infty} e^{-x^2}\,\mathrm{erf}\,(x)\,dx &= \frac{\sqrt{\pi}}{4} \\
\int_0^{\infty} e^{-x^2}\,\mathrm{erf}^{\,3}(x)\,dx &= \frac{\sqrt{\pi}}{8} \\
\int e^{-x^2}\,\mathrm{erf}^{\,n}(x)\,dx &= \frac{\sqrt{\pi}\,\mathrm{erf}^{\,n+1}(x)}{2(n+1)} \\
\int_0^{\infty} x\,e^{-\alpha^2x^2}\,\mathrm{erf}\,(\beta x)\,dx &= \frac{\beta}{2\alpha^2\sqrt{\alpha^2+\beta^2}} \\
\int \mathrm{erf}^{\,2}(x)dx &= \frac{\sqrt{\pi}\,x\,\mathrm{erf}^{\,2}x+2\,e^{-x^2}\,\mathrm{erf}\,(x)}{\sqrt{\pi}} - \frac{2\,\mathrm{erf}\left(\sqrt{2}\,x\right)}{\sqrt{2\pi}}
\end{aligned}$$

3 $\mathbf{Pr}[x_1 < x_2]$

Let $x_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $x_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$ be two independent Gaussian RVs. I would like to calculate $\Pr[x_1 < x_2]$. This is given by

$$\begin{aligned}
\Pr[x_1 < x_2] &= \int_{-\infty}^{\infty} \int_{x_1}^{\infty} f_{\mu_1,\sigma_1^2}(x_1) f_{\mu_2,\sigma_2^2}(x_2)\, dx_2 dx_1 \\
&= \frac{1}{2} - \frac{1}{2\sigma_2\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[\frac{-(x_1-\mu_1)^2}{2\sigma_1^2}\right] \operatorname{erf}\left[\frac{x_1-\mu_2}{\sqrt{2}\sigma_2}\right] dx_1 \\
&= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{\mu_2-\mu_1}{\sqrt{2(\sigma_1^2+\sigma_2^2)}}\right)\right]
\end{aligned}$$

using the first integral of the last section.

The same result can be obtained more easily - the sum of two Gaussians is Gaussian, so the result is immediate from $\Pr[x_1 < x_2] = \Pr[x_2 - x_1 > 0]$.

4 The mean of the maximum of Gaussians

I would like the compute the mean of the maximum of $n = 2, 3, 4, \ldots$ iid N(0,1) variates. This is given by

$$\mu(n) = n \int_{-\infty}^{\infty} x\, f(x)\, F^n(x)\, dx = \frac{n 2^{3-n}}{\sqrt{2\pi}} \int_0^{\infty} u\, e^{-u^2} \sum_{k=1, k \text{ odd}}^{n-1} \binom{n-1}{k} \operatorname{erf}^k(u)\, du.$$

We have $\mu(2) = 1/\sqrt{\pi}$, $\mu(3) = 3/(2\sqrt{\pi})$, and it seems that $\cos(\mu(4)\pi^{3/2}) = 23/27$, though I cannot prove this.

Using results from David *Order statistics* pp. 32-38:

$$
\begin{aligned}
\mu(1,1) &= 0 \\
\mu(1,2) &= -\pi^{-1/2} \\
\mu(2,2) &= \pi^{-1/2} \\
\mu(1,3) &= -\frac{3}{2}\pi^{-1/2} \\
\mu(2,3) &= 0 \\
\mu(3,3) &= \frac{3}{2}\pi^{-1/2} \\
\mu(1,4) &= -6\pi^{-3/2}\arctan(2^{1/2}) \\
\mu(2,4) &= ? \\
\mu(3,4) &= ? \\
\mu(4,4) &= -\mu(1,4) \\
\mu(1,5) &= -\mu(5,5) \\
\mu(2,5) &= -\mu(4,5) \\
\mu(3,5) &= 0 \\
\mu(4,5) &= \frac{5}{4}\pi^{-1/2} - 15\pi^{-3/2}\arcsin(1/3) \\
\mu(5,5) &= \frac{5}{4}\pi^{-1/2} + \frac{15}{2}\pi^{-3/2}\arcsin(1/3)
\end{aligned}
$$

Here $\mu(r,n)$ is the expectation value of the rth order statistic for a sample of n N(0,1) RVs.