Graph models of wireless networks

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University of Essex Maths 2009-02-05 1400
Corrected version 2009-02-06
Wireless networks
PPPP(\(\lambda\)): definitions and statistical properties

- PPPP(\(\lambda\)): planar Poisson point process

\[ f_k(x) = 2(\frac{\lambda}{\pi})^k \Gamma(k) \exp(-\frac{\lambda \pi x^2}{2}) x^{2k-1} \]

- Mean distances are \(\frac{1}{2} \lambda - \frac{1}{2}, \frac{3}{4} \lambda - \frac{1}{2}, \frac{15}{16} \lambda - \frac{1}{2}, \ldots\)

- Also binomial PPP

- Also nonhomogeneous case
PPPP($\lambda$): definitions and statistical properties

- PPPP($\lambda$): planar Poisson point process
  - region $R$ has Poi($\lambda||R||$) points

- pdf of distance $X$ to $k$th nearest neighbour ($k=1,2,3,...$) is
  $$f_k(x) = 2(\frac{\lambda}{\pi})^k \Gamma(k) \exp(-\frac{\lambda}{\pi}x^2)x^{2k-1}$$

- mean distances are $\frac{1}{2}\lambda - \frac{1}{2}$, $\frac{3}{4}\lambda - \frac{1}{2}$, $\frac{15}{16}\lambda - \frac{1}{2}$, ...

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  - region $R$ has $\text{Poi}(\lambda\|R\|)$ points
  - numbers in non-overlapping regions are independent

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PPPP($\lambda$): radial generation

$\triangleright$ $s = 0$

$\triangleright$ do

$s \leftarrow s - \log(\text{Uniform}(0, 1))$

$\theta = 2\pi \text{Uniform}(0, 1)$

$r = \sqrt{s / (\pi \lambda)}$

$x = r \cos \theta$

$y = r \sin \theta$

$\triangleright$ while $r < \text{desired maximum radius}$
GRG($\lambda, \rho$): definitions and statistical properties

- take PPPP($\lambda$); call them nodes
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- connect nodes separated by less than \(\rho\) by \textit{links} or \textit{edges}
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- surprisingly, the degree-degree correlation is the same, independently of $\lambda$ and $\rho$!
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GRG(20, $\rho$) degree-degree distribution

$\rho=0.3$

$\rho=0.5$
If $X_0 \sim \text{Poi}(\lambda_0)$, $X_1 \sim \text{Poi}(\lambda_1)$, $X_2 \sim \text{Poi}(\lambda_1)$ are independent, and $Y_1 = X_1 + X_0$, $Y_2 = X_2 + X_0$, then

$$\text{corr}(Y_1, Y_2) = \frac{\lambda_0}{\lambda_0 + \lambda_1}$$
GRG(λ, ρ) degree-degree correlation

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  \[
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  \]

- For PPPP, degree-degree correlation is $E[\text{corr}]

\[
= \int_0^\rho \frac{2\rho^2 \arccos(x/(2\rho)) - (x/2)\sqrt{4\rho^2 - x^2}}{\pi \rho^2} \frac{2x}{\rho^2} \, dx
\]

\[
= 1 - 3\sqrt{3}/(4\pi) \approx 0.5865
\]
GRG($\lambda, \rho$) degree-degree correlation - square

exact (doable but messy); simulation
GRG($\lambda, \rho, \text{unit circle}$): degree distribution

- pdf of distance of a random point from the centre, given that it is within $1 - \rho$ of the edge:

$$f_\rho(x) = \frac{(4 - 2\rho)x + 2\rho - 2}{\rho^3 - 2\rho^2 + 2\rho} [1 - \rho < x < 1]$$
GRG($\lambda, \rho$, unit circle): degree distribution

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- area of overlap of circles radius 1 and $\rho$, centres $x$ apart:

$$A(x) = \rho^2 \arccos \left( \frac{x^2 + \rho^2 - 1}{2x\rho} \right) + \arccos \left( \frac{x^2 - \rho^2 + 1}{2x} \right) - \frac{1}{2} \left[ (1-x+\rho)(x+\rho-1)(x-\rho+1)(x+\rho+1) \right]^{1/2}$$
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\]

- Prob[\(d=k\)]=

\[
(1-\rho)^2 \text{Poi}(A(x), k) + \rho(2-\rho) \int_{1-\rho}^{1} \text{Poi}(A(x)\lambda) f_\rho(x) \, dx
\]

- where \(\text{Poi}(\mu, k) = e^{-\mu} \mu^k / k!\)
GRG($\lambda$, $\rho$, unit circle) degree distribution

![Graph showing degree distribution with exact, simulation, and Poisson approximations, indicating edge effect.](image-url)
Poisson maxima 1

\(\{X_1, X_2, \ldots, X_n\}\) iid, \(\Pr[X_i = k] = e^{-\lambda} \frac{\lambda^k}{k!}\)
Poisson maxima 1

- $\{X_1, X_2, \ldots, X_n\}$ iid, $\Pr[X_i=k]=e^{-\lambda} \frac{\lambda^k}{k!}$
- $M_n=\max(X_i)$
\[ \{X_1, X_2, \ldots, X_n\} \text{ iid, } \Pr[X_i = k] = e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ M_n = \max(X_i) \]

The distribution of the maximum of Poisson variables for \( \lambda = 1/2, 1, 2, 5 \) (left to right) and \( n = 10^0, 10^2, 10^4, \ldots, 10^{24} \)
Poisson maxima 2

- Anderson: $\exists I_n \in \mathbb{Z}$ s.t. $\Pr[M_n \in \{I_n, I_n+1\}] \to 1$
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- Kimber: \( I_n \sim \log n / \log \log n \) as \( n \to \infty \)
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The maximal probability (with respect to \( I_n \)) that \( M_n \in \{I_n, I_n+1\} \) for \( \lambda = 1/2, 1, 2, 5 \) (left to right) and \( 10^0 \leq n \leq 10^{40} \). The curves show the probability that \( M_n \) takes either of its two most frequently occurring values.
Poisson maxima 3

\[ M_n \sim x_0 \equiv \log n / W \left( \frac{\log n}{e \lambda} \right) \]
Poisson maxima 3

- $M_n \sim x_0 \equiv \log n/W \left( \frac{\log n}{e \lambda} \right)$
- $M_n \sim x_1 = x_0 + \frac{\log \lambda - \lambda - \log(2\pi)/2 - 3 \log(x_0)/2}{\log(x_0) - \log \lambda}$
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Roman networks
Anglo-Saxon networks