Optimal trip planning in transport systems with random delays

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Optimizing departure time

The primary problem:

- Given:
  1. The time a passenger would like to reach the destination
  2. The probability that a passenger would like to reach the destination on time (a high probability if it is a time-critical trip or a smaller probability if it is less important)
- Find:
  1. The best route
  2. The latest time the passenger should start from the first train station

Let

- τ be the time the passenger would like to reach the destination
- ϵ be the probability that the passenger would like to reach the destination "on time"
- t_d be the time the passenger departs from the first station
- T_a be the time the passenger arrives at the destination (arrival time)

Then the optimization problem is:

$$\begin{array}{rl} \text{max} & t_d \\ \text{subject to} & \text{Prob}[T_a > \tau] < 1 - \epsilon \end{array}$$

Modelling train delay

In order to calculate Prob$[T_a > \tau]$, we will need to solve the following secondary problem:

- Given:
  1. A route
  2. A timetabled service for each train
  3. A model of the distribution of delays for each train
  4. A model of the distribution of starting time of the passenger
- Find:
  1. The probability distribution of the arrival time T_a for the passenger at the destination

Assumptions and notations

Assumptions:

- The departure times of any two trains are statistically independent
- The order of the departure for trains may vary due to delay
- When changing trains, passengers always catch the first train that departs to their next station on their chosen route

Notation:

$X_i \sim f_i$ probability distribution function (pdf)

F_i cumulative distribution function (cdf)

$$F_i(t) = \int_{-\infty}^{t} f_i(x)dx$$

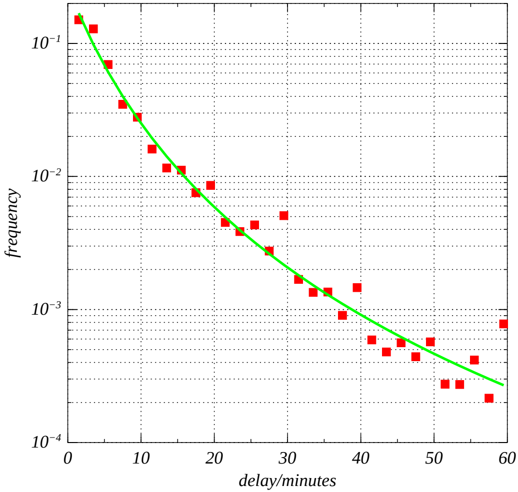
$$[\![x \geqslant t]\!] = \begin{cases} 1 & \text{if } x \geqslant t, \\ 0 & \text{otherwise} \end{cases}$$

The q-exponential law

- Exponential law: $f_\beta(x) \propto \exp(-\beta x)$
- $e_{r,\beta}(x) := Z(1+\frac{\beta x}{r})^{-r}$, $\beta > 0$, $r > 1$
- $Z := \beta(1-\frac{1}{r})$
- mean $\mu := \frac{1}{\beta(1-\frac{2}{r})}$
- $\lim_{r\to\infty} e_{r,\beta}(x) = Z\exp(-\beta x)$
- small r gives a power-law (long tail)
- The departure times for every trains can be modelled by q-exponential distribution by some parameters β and r.

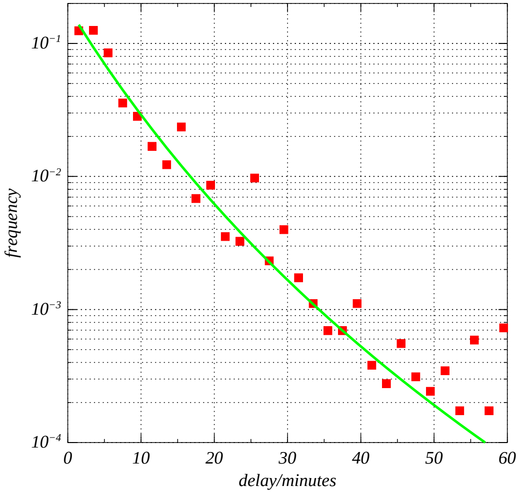
Birmingham, all departures

$r = 3.8$
$\beta = 0.33$



Coventry to Birmingham

$r = 9.0$
$\beta = 0.21$



The discrete $q-$exponential model

- The discrete $q-$exponential model is more suited for the train model since the departure times of trains are considered in minutes
- Bins have length $\mathrm{d}t$, typically one minute
- The values for each bin are calculated using the CDF of the continuous model, i.e. $\int_a^{a+\mathrm{d}t} e_{r,\beta}(x)\ \mathrm{d}x$ where a is the value of the bin
- The distribution is truncated; the departure delay cannot be <0 or greater than some maximum

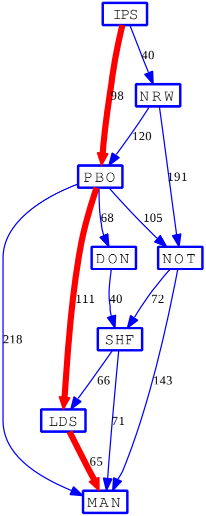
Stochastically short paths

- Given a graph with RVs as edge weights and two nodes, we could:
  - minimize expected time to travel between the nodes
  - find a route which maximizes the probability that it is shortest
  - find the route of shortest mean time, subject to some condition on the variance
  - ...

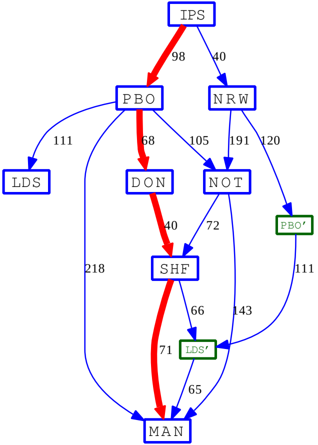
The problem formalized

- Given: a weighted digraph g, a timetable $\mathsf{TT}(n_0, n_1)$ for each arc $(n_0, n_1) \in g$, a final arrival time T_a, and parameters $\tau > 0, \epsilon > 0$.
- To find: a route ρ and maximal departure time t such that Prob[arrival after $\alpha + \tau$] $< \epsilon$

Short paths in a weighted digraph

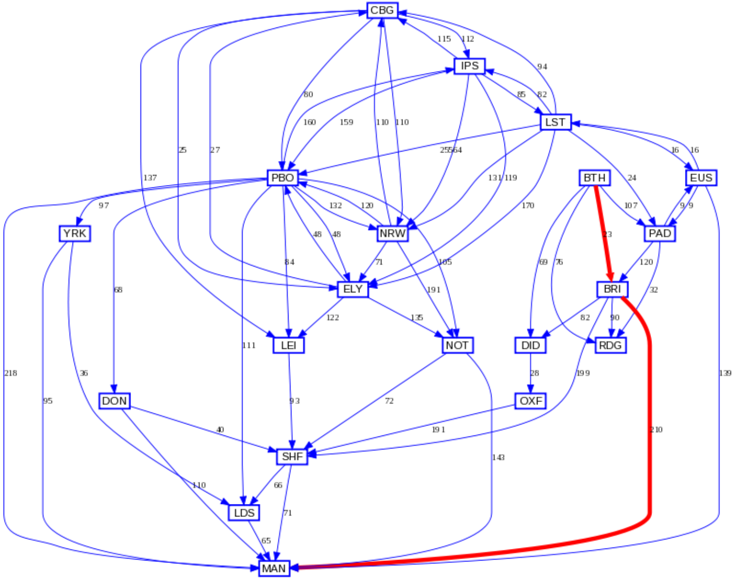


time=275



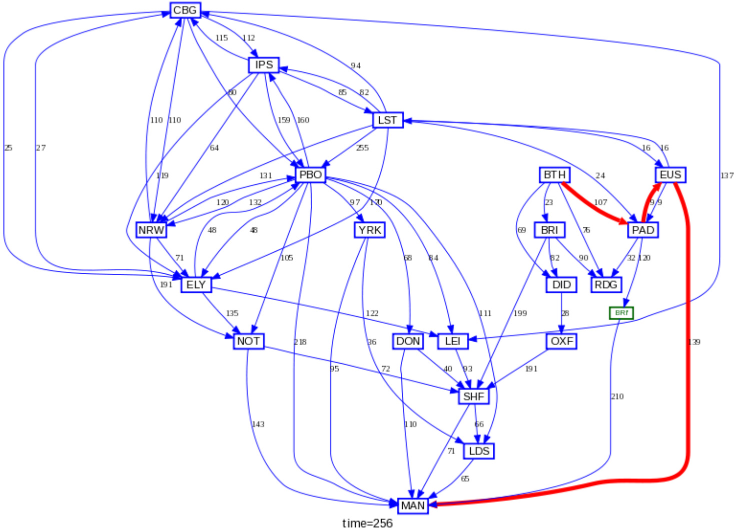
time=278

Bath to Manchester, shortest mean time



time=233

Bath to Manchester, second shortest mean time



Transformation using kernel

- The departure time of the passenger at the initial station is modelled as a probability distribution
- We need to compute the probability distribution model for the arrival time of the passenger at the next station

$$A_0 \xrightarrow[K_0]{} A_1 \xrightarrow[K_1]{} \ldots \xrightarrow[K_{r-1}]{} A_r$$

A_i is the probability distribution of arrival time i.

K_i represents the set of probability distributions of departure times of each train at station i and we will call it the *kernel*.

Aside: order statistics

Let $X_i \sim f$, $i=0,\ldots,n-1$ be iid. Define $X_{(0)}$ to be the minimum X_i. We compute the pdf $f_{(0)}$ of $X_{(0)}$:

$$\begin{aligned}
f_{(0)}(x)\mathrm{d}x &= \mathrm{Prob}[X_{(0)} \in \mathrm{d}x] \\
&= \mathrm{Prob}[\text{one } X_i \in \mathrm{d}x,\ \text{others} > x] \\
&= \sum_i \mathrm{Prob}[X_i \in \mathrm{d}x, \underset{j\neq i}{X_j} > x] \\
&= \sum_i \mathrm{Prob}[X_i \in \mathrm{d}x]\mathrm{Prob}[\underset{j\neq i}{X_j} > x] \\
f_{(0)}(x) &= \sum_i f(x) \prod_{j\neq i} (1-F(x)) \\
&= nf(x)(1-F(x))^{n-1}
\end{aligned}$$

The continuous time model

Let $K_{(0|t)}(x)$ be the probability density of departing at time x given that the passenger arrives at time t.

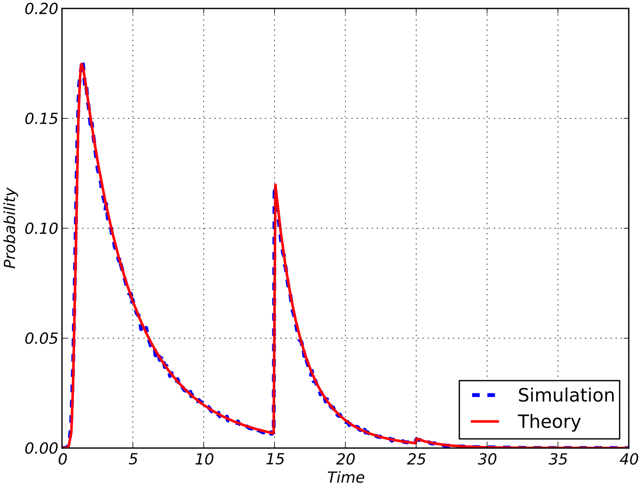
$$
\begin{aligned}
K_{(0|t)}(x)\mathrm{d}x &= [\![x>t]\!]\,\mathsf{Prob}[X_{(0|t)}\in \mathrm{d}x] \\
&= [\![x>t]\!]\,\mathsf{Prob}[\text{one } X_i\in \mathrm{d}x,\ \text{others } >x \text{ or } \leqslant t] \\
&= [\![x>t]\!]\sum_i \mathsf{Prob}[X_i\in \mathrm{d}x, \underset{j\neq i}{X_j} > x \text{ or } \leqslant t] \\
&= [\![x>t]\!]\sum_i \mathsf{Prob}[X_i\in \mathrm{d}x]\,\mathsf{Prob}[\underset{j\neq i}{X_j} > x \text{ or } \leqslant t] \\
K_{(0|t)}(x) &= [\![x>t]\!]\sum_i f_i(x)\prod_{j\neq i}(1-F_j(x)+F_j(t))
\end{aligned}
$$

Integral transforms

$$A_n(x) = \int_{-\infty}^{\infty} A_{n-1}(t) K_{(n-1|t)}(x) \mathrm{d}t, \; n = 1, 2, \ldots$$

- Exact results (and agree with simulation)
- Integrals cannot be done analytically
- The amount of computer time needed to calculate $A_i(x)$ increases dramatically as i increases

Continuous model simulation for catching one train



The discrete time model

Let T be the set of all trains, t be the time the passenger arrives at the station and x be the time the passenger departs.

$$\begin{aligned} K_{(0|t)}(x) &= [\![x>t]\!]\,\mathrm{Prob}[X_{(0|t)}=x] \\ &= [\![x>t]\!]\,\mathrm{Prob}[\text{at least one } X_i = x,\ \text{others } > x \text{ or } \leqslant t] \end{aligned}$$

Let T be the set of trains departing after time t; then

$$\begin{aligned} K_{(0|t)}(x) &= [\![x>t]\!] \sum_{\emptyset\neq S\subseteq T} \mathrm{Prob}[\underset{i\in S}{X_i}=x,\ \underset{j\notin S}{X_j}>x \text{ or } \leqslant t] \\ &= [\![x>t]\!] \sum_{\emptyset\neq S\subseteq T} \mathrm{Prob}[\underset{i\in S}{X_i}=x]\,\mathrm{Prob}[\underset{j\notin S}{X_j}>x \text{ or } \leqslant t] \\ &= [\![x>t]\!] \sum_{\emptyset\neq S\subseteq T} \prod_{i\in S} f_i(x) \prod_{j\notin S} (1-F_j(x)+F_j(t)) \end{aligned}$$

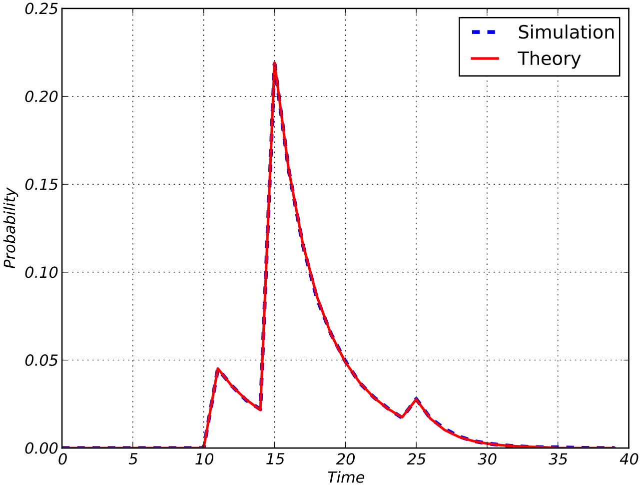
Matrix multiplication

Let $K_0 = k_{ij}^{(0)} = (K_{(0|j)}(i))$, then

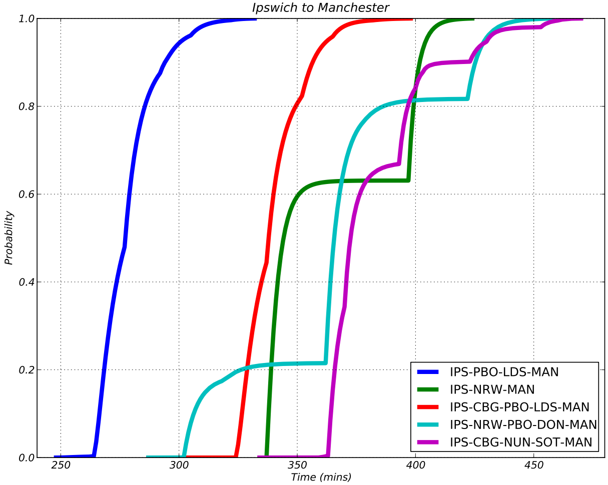
$$
\begin{aligned}
A_1 &= K_0 A_0 \\
A_2 &= K_1 A_1 = K_1 K_0 A_0 \\
&\dots \\
A_n &= K_{n-1} \dots K_0 A_0
\end{aligned}
$$

- Kernel and the distribution of arrival time can be considered as matrix and vector
- Finding the distribution of the arrival time at the next station is exactly the same as matrix multiplication
- Matrix calculations are generally much quicker than integration

Discrete model simulation for catching one train



CDF for journey times from Ipswich to Manchester



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