Graph models of wireless networks

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NET-ACE Brunel University 2008 Sep 19 0930
Wireless networks
PPPP(λ): definitions and statistical properties

- PPPP(λ): planar Poisson point process

A region $R$ has $\text{Poi}(λ|R)$ points. The numbers in non-overlapping regions are independent.

The pdf of distance $X$ to the $k$th nearest neighbour ($k = 1, 2, 3, \ldots$) is

$$f_k(x) = 2\left(\frac{\lambda}{\pi}\right)^k \Gamma(k) \exp\left(-\frac{\lambda}{\pi}x^2\right) x^{2k-1}$$

Mean distances are $k\text{th}$-order mean distances.

Also binomial PPP also nonhomogeneous case.
PPPP(\(\lambda\)): definitions and statistical properties

- PPPP(\(\lambda\)): planar Poisson point process
  - region \(R\) has Poi(\(\lambda \parallel R\parallel\)) points

\[
f_k(x) = 2(\frac{\lambda}{\pi})^k \Gamma(k) \exp(-\lambda \pi x^2) x^{2k-1}
\]

- Mean distances are
  \[
  \frac{1}{2} \lambda - \frac{1}{2}, \frac{3}{4} \lambda - \frac{1}{2}, \frac{15}{16} \lambda - \frac{1}{2}, \ldots
  \]

- Also binomial PPP
- Also nonhomogeneous case
PPPP(\(\lambda\)): planar Poisson point process

- region \(R\) has Poi(\(\lambda\|R\|\)) points
- numbers in non-overlapping regions are independent
PPPP(\(\lambda\)): planar Poisson point process

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- numbers in non-overlapping regions are *independent*
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  f_k(x) = \frac{2(\lambda \pi)^k}{\Gamma(k)} \exp(-\lambda \pi x^2) x^{2k-1}
  \]  

(mean distances are \(\frac{1}{2} \frac{1}{\lambda - \frac{1}{2}}, \frac{3}{4} \frac{1}{\lambda - \frac{1}{2}}, \frac{15}{16} \frac{1}{\lambda - \frac{1}{2}}, \ldots\) )
PPPP(\(\lambda\)): definitions and statistical properties

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  - region \(R\) has Poi(\(\lambda\|R\|\)) points
  - numbers in non-overlapping regions are \textit{independent}
  - pdf of distance \(X\) to \(k\)th nearest neighbour (\(k=1, 2, 3, \ldots\)) is (Haenggi)
    \[
    f_k(x) = \frac{2(\lambda\pi)^k}{\Gamma(k)} \exp(-\lambda\pi x^2) x^{2k-1}
    \]
  - mean distances are \(\frac{1}{2} \lambda^{-1/2}, \frac{3}{4} \lambda^{-1/2}, \frac{15}{16} \lambda^{-1/2}, \ldots\)

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PPPP(\(\lambda\)): radial generation

\[ s = 0 \]

\[ \text{do} \]

\[ s \leftarrow s - \log(\text{Uniform}(0, 1)) \]
\[ \theta = 2\pi \text{Uniform}(0, 1) \]
\[ r = \sqrt{s / (\pi \lambda)} \]
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ \text{while } r < \text{desired maximum radius} \]
GRG(\(\lambda, \rho\)): definitions and statistical properties

- take PPPP(\(\lambda\)); call them \textit{nodes}
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- connect nodes separated by less than \(\rho\) by *links* or *edges*
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- degree of a point (not necessarily a node) is defined as the number of Poisson points in its disk
- distribution of degree is Poisson

\([D_0, D_1]\) of an edge is the degree of its end nodes

\(-\frac{3}{4}\sqrt{\frac{3}{\pi}}\)

Surprisingly, the degree-degree correlation is the same, independent of \(\lambda\) and \(\rho\)!
GRG($\lambda, \rho$): definitions and statistical properties

- take PPPP($\lambda$); call them *nodes*
- connect nodes separated by less than $\rho$ by *links* or *edges*
- degree of a point (not necessarily a node) is defined as the number of Poisson points in its disk
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- degree ($D_0, D_1$) of an edge is the degree of its end nodes
GRG(\(\lambda, \rho\)): definitions and statistical properties

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- *degree \((D_0, D_1)\) of an edge* is the degree of its end nodes
- the cluster coefficient is \(1 - \frac{3\sqrt{3}}{4\pi}\)
GRG(λ, ρ): definitions and statistical properties

- take PPPP(λ); call them *nodes*
- connect nodes separated by less than ρ by *links* or *edges*
- degree of a point (not necessarily a node) is defined as the number of Poisson points in its disk
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- degree \((D_0, D_1)\) of an edge is the degree of its end nodes
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- surprisingly, the degree-degree correlation is the same, independent of λ and ρ!
GRG(20, $\rho$) degree-degree distribution

$\rho = 0.3$

$\rho = 0.5$
GRG(\(\lambda, \rho\)) degree-degree correlation

- If \(X_0 \sim \text{Poi}(\lambda_0)\), \(X_1 \sim \text{Poi}(\lambda_1)\), \(X_2 \sim \text{Poi}(\lambda_1)\) are independent, and \(Y_1 = X_1 + X_0\), \(Y_2 = X_2 + X_0\), then

\[
\text{corr}(Y_1, Y_2) = \frac{\lambda_0}{\lambda_0 + \lambda_1}
\]
GRG($\lambda, \rho$) degree-degree correlation

If $X_0 \sim \text{Poi}(\lambda_0)$, $X_1 \sim \text{Poi}(\lambda_1)$, $X_2 \sim \text{Poi}(\lambda_1)$ are independent, and $Y_1 = X_1 + X_0$, $Y_2 = X_2 + X_0$, then

$$\text{corr}(Y_1, Y_2) = \frac{\lambda_0}{\lambda_0 + \lambda_1}$$

For PPPP, degree-degree correlation is $E[\text{corr}]$

$$= \int_0^\rho \frac{2\rho^2 \arccos(x/(2\rho)) - (x/2)\sqrt{4\rho^2 - x^2}}{\pi \rho^2} \frac{2x}{\rho^2} \, dx$$

$$= 1 - 3\sqrt{3}/(4\pi) \approx 0.5865$$
GRG($\lambda, \rho$) degree-degree correlation - square

exact (doable but messy); simulation
GRG(\(\lambda, \rho\), unit circle): degree distribution

- pdf of distance of a random point from the centre, given that it is within \(1 - \rho\) of the edge:

\[
f_\rho(x) = \frac{(4 - 2\rho)x + 2\rho - 2}{\rho^3 - 2\rho^2 + 2\rho} \quad [1 - \rho < x < 1]
\]
GRG(\(\lambda, \rho\), unit circle): degree distribution

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f_\rho(x) = \frac{(4-2\rho)x + 2\rho - 2}{\rho^3 - 2\rho^2 + 2\rho} \quad \text{for} \quad [1-\rho<x<1]\]

- area of overlap of circles radius 1 and \(\rho\), centres \(x\) apart:

\[
A(x) = \rho^2 \arccos \left( \frac{x^2 + \rho^2 - 1}{2x\rho} \right) + \arccos \left( \frac{x^2 - \rho^2 + 1}{2x} \right) - \frac{1}{2} \left[ (1-x+\rho)(x+\rho-1)(x-\rho+1)(x+\rho+1) \right]^{1/2}
\]
GRG($\lambda, \rho$, unit circle): degree distribution

- pdf of distance of a random point from the centre, given that it is within $1-\rho$ of the edge:
  \[ f_{\rho}(x) = \frac{(4 - 2\rho)x + 2\rho - 2}{\rho^3 - 2\rho^2 + 2\rho} \left[ 1 - \rho < x < 1 \right] \]

- area of overlap of circles radius 1 and $\rho$, centres $x$ apart:
  \[ A(x) = \rho^2 \arccos \left( \frac{x^2 + \rho^2 - 1}{2x\rho} \right) + \arccos \left( \frac{x^2 - \rho^2 + 1}{2x} \right) - \frac{1}{2} \left[ (1-x+\rho)(x+\rho-1)(x-\rho+1)(x+\rho+1) \right]^{1/2} \]

- Prob[$d=k$] =
  \[ (1-\rho)^2 \text{Poi}(A(x), k) + \rho(2-\rho) \int_{1-\rho}^{1} \text{Poi}(A(x)\lambda) \, f_{\rho}(x) \, dx \]
  where $\text{Poi}(\mu, k) = e^{-\mu} \frac{\mu^k}{k!}$
GRG($\lambda$, $\rho$, unit circle) degree distribution

exact; simulation; Poisson - ignores edge effect.
Poisson maxima 1

\[
\{X_1, X_2, \ldots, X_n\} \text{ iid, } \Pr[X_i = k] = e^{-\lambda} \frac{\lambda^k}{k!}
\]
Poisson maxima 1

- \( \{X_1, X_2, \ldots, X_n\} \) iid, \( \Pr[X_i = k] = e^{-\lambda} \lambda^k / k! \)
- \( M_n = \max(X_i) \)
\( \{X_1, X_2, \ldots, X_n\} \) iid, \( \Pr[X_i = k] = e^{-\lambda} \frac{\lambda^k}{k!} \)

\( M_n = \max(X_i) \)

The distribution of the maximum of Poisson variables for \( \lambda = \frac{1}{2}, 1, 2, 5 \) (left to right) and \( n = 10^0, 10^2, 10^4, \ldots, 10^{24} \)
Poisson maxima 2

- Anderson: \( \exists I_n \in \mathbb{Z} \) s.t. \( \Pr M_n \in (I_n, I_{n+1}) \to 1 \)
Poisson maxima 2

- Anderson: \( \exists I_n \in \mathbb{Z} \text{ s.t. } \Pr M_n \in (I_n, I_{n+1}) \rightarrow 1 \)
- Kimber: \( I_n \sim \log n / \log \log n \text{ as } n \to \infty \)
Poisson maxima 2

- Anderson: \( \exists I_n \in \mathbb{Z} \text{ s.t. } \Pr M_n \in (I_n, I_{n+1}) \to 1 \)
- Kimber: \( I_n \sim \log n / \log \log n \) as \( n \to \infty \)

The maximal probability (with respect to \( I_n \)) that \( M_n \in \{ I_n, I_{n+1} \} \) for \( \lambda = 1/2, 1, 2, 5 \) (left to right) and \( 10^0 \leq n \leq 10^{40} \). The curves show the probability that \( M_n \) takes either of its two most frequently occurring values.
- $M_n \sim x_0 \equiv \log n/W \left( \frac{\log n}{e \lambda} \right)$
Poisson maxima 3

\[
M_n \sim x_0 \equiv \log n/W \left( \frac{\log n}{e \lambda} \right)
\]

\[
M_n \sim x_1 = x_0 + \frac{\log \lambda - \lambda - \log(2\pi)/2 - 3\log(x_0)/2}{\log(x_0) - \log \lambda}
\]
Poisson maxima 3

\[ M_n \sim x_0 \equiv \log n / W \left( \frac{\log n}{e \lambda} \right) \]

\[ M_n \sim x_1 = x_0 + \frac{\log \lambda - \lambda - \log(2\pi)/2 - 3 \log(x_0)/2}{\log(x_0) - \log \lambda} \]
Chromatic number
Chromatic number
Objectives

- Optimization problem:

\[ \min_x f(x) \]

- Minimizing the average interference:

\[ f(x) = \frac{1}{n} \sum_i I_i(x) \]

- Minimizing the maximum interference:

\[ f(x) = \max_i I_i(x) \]
Optimization results

Modulation schemes: 11Mbps, 5.5Mbps, 2Mbps, 1Mbps

(a) 20 APs using the same power level and channel
(b) 20 APs with randomly assigned channels
(c) 20 APs using the same power level, but with an optimized channel allocation (13 channels)