The work of George Szekeres on functional equations

Keith Briggs
Keith.Briggs@bt.com
http://keithbriggs.info

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George (György) Szekeres 1911-2005

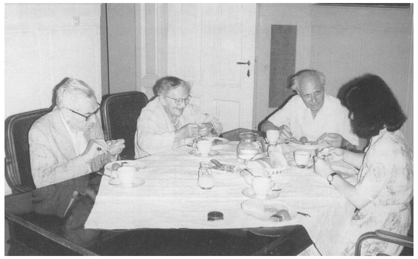
George Szekeres

- ⋆ studied chemical engineering at Technological University of Budapest
- ⋆ refugee in Shanghai 1940s
- ⋆ Adelaide 1948-63
- ⋆ UNSW 1963-75
- ⋆ died Adelaide 2005 Aug 28

The work of George Szekeres

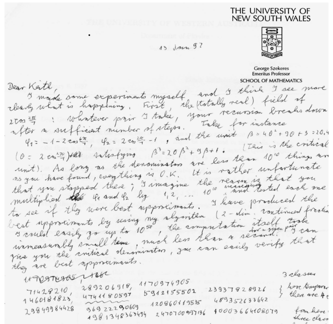
- ⋆ first co-author of Erdős
- ⋆ graph theory
- ⋆ general relativity
- ⋆ functional equations
- ⋆ multi-dimensional continued fractions
- ⋆ lots more . . .

Szekeres and colleagues



Paul Erdős, Esther Klein, George Szekeres, Fan Chung

A letter from George



Outline

- ⋆ analytic iteration
- ⋆ Schröder & Koenigs
- ⋆ Szerekes on the Schröder and Abel equations
- ⋆ Szerekes on the Feigenbaum functional equation
- ⋆ Szerekes on Abel's equation and growth rates
- ⋆ formal iteration & Julia's equation (my speculations)
- ⋆ Jacobian conjecture (my speculations)

Iteration theory and functional equations

- ⋆ map $f:\mathbb{X}\to\mathbb{X}$
- ⋆ orbit $x_n=f(x_{n-1})\equiv f^{<n>}(x_0)$, $n=1,2,3,\dots$
- ⋆ around 1870 Schröder proposed studying the orbit by trying to find a new coordinate system in which the orbit 'looks simpler'
- ⋆ simplest case: explicit iterability
- ⋆ $\sigma\circ f(x)-f'(0)\sigma(x)=0 \quad \forall x$
- ⋆ $\sigma\circ f^{<n>}(x)=(f'(0))^n\sigma(x), \quad n=0,1,2,\dots$

Friedrich Wilhelm Karl Ernst Schröder 1841-1902

- ⋆ Pforzheim
- ⋆ *Ueber iterirte Functionen* Math. Annalen **3** 296-322 (1871)

Solutions of the Schröder equation

- ⋆ Schröder found several explicit solutions to his equation in terms of elementary functions
- ⋆ $f_1(x)=\quad 2(x+x^2)\Rightarrow\sigma(x)=\log(1+2x)/2$
- ⋆ $f_2(x)=-2(x+x^2)\Rightarrow\sigma(x)=\sqrt{3}/2(\arccos(-1/2-x)-2\pi/3)$
- ⋆ $f_3(x)=\quad 4(x+x^2)\Rightarrow\sigma(x)=(\operatorname{arcsinh}(\sqrt{x}))^2$
- ⋆ cases 2 and 3 are conjugate with respect to the function $h(x)=-3/2-2x$. That is, $h\circ f_3=f_2\circ h$

The Schröder equation

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\quad f\quad} & \mathbb{X} \\ \sigma\downarrow & & \downarrow\sigma \\ \mathbb{Y} & \xrightarrow[x\mapsto f_1 x]{} & \mathbb{Y} \end{array}$$

- ⋆ $\sigma\circ f(x)=f_1\sigma(x)$ (Schröder, $f_1\equiv f'(0)$)
- ⋆ $\alpha\circ f(x)=\alpha(x)+1$ (Abel)
- ⋆ $\beta\circ f(x)=(\beta(x))^2$ (Boettcher=Бётхеръ)
- ⋆ $\iota\circ f(x)=f'(x)\iota(x)$ (Julia)

Formal solution of the Schröder equation

- ⋆ for $f(z)=\sum_{i=1}^{\infty} f_i z^i$ and $\sigma(z)=\sum_{i=1}^{\infty}\sigma_i z^i$ we have

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ f_2 & f_1^2-f_1 & 0 & \cdots \\ f_3 & 3f_1f_2 & f_1^3-f_1 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \vdots\end{bmatrix}=\begin{bmatrix}0\\0\\0\\ \vdots\end{bmatrix}$$

- ⋆ defn: family of (continuous) iterates: $f^{<s>}(z)=\sigma^{<-1>}(f_1^s\sigma(z))$
- ⋆ obtain formal solution from $f^{<s>}\circ f=f\circ f^{<s>}$

Schröder, Abel, Boettcher, and Julia

$$\begin{tikzcd} \mathbb{X} \arrow[r, "f"] \arrow[d, "\sigma"'] & \mathbb{X} \arrow[d, "\sigma"] \\ \mathbb{X} \arrow[r, "x \mapsto f_1 x"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X} \arrow[r, "f"] \arrow[d, "\alpha"'] & \mathbb{X} \arrow[d, "\alpha"] \\ \mathbb{X} \arrow[r, "x \mapsto x+1"'] & \mathbb{X} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{X} \arrow[r, "f"] \arrow[d, "\beta"'] & \mathbb{X} \arrow[d, "\beta"] \\ \mathbb{X} \arrow[r, "x \mapsto x^2"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{X} \arrow[r, "f"] \arrow[d, "\iota"'] & \mathbb{X} \arrow[d, "\iota"] \\ \mathbb{X} \arrow[r, "x \mapsto f'(x)\, x"'] & \mathbb{X} \end{tikzcd}$$

Gabriel Koenigs 1858-1931

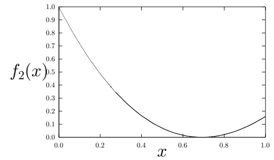
- let $f(z) = \sum_{i=1}^{\infty} f_i\, z^i$ be analytic with $|f_1| < 1$
- then the Schröder equation has an analytic solution $\sigma(z) = \sum_{i=1}^{\infty} \sigma_i\, z^i$, where each σ_k depends only on f_i for $i \leqslant k$
- also $\kappa(z) \equiv \lim_{i\to\infty} f_1^{-i} f^{<i>}(z)$ exists and satisfies $\kappa \circ f(z) = f_1 \kappa(z)$
- Kneser: sufficient to have $f(z) = f_1\, z + \mathcal{O}(|z|^{1+\delta})$, $\delta > 0$ as $z \to 0$
- Szekeres [1]: for $f(z) = \frac{z}{2} + \frac{z^2}{3\pi} \sin\left(\frac{\pi}{|z|}\right)$, σ has 'flat spots'
- Szekeres: if $f : \mathbb{R} \to \mathbb{R}$ is continuous, strictly monotone increasing, $f'(x)$ exists and $f'(x) = a + \mathcal{O}(x^\delta)$, then the Koenigs function κ exists and is invertible

Solution of the Schröder equation [1]

- cases for f:
  - *$f(z) = \sum_{i=m}^{\infty} f_i\, z^i$ analytic, $m > 0$*
  - *$f(z) \sim \sum_{i=m}^{\infty} f_i\, z^i$ as $z \to 0$, $m > 0$*
  - *f a continuous real function*
- case $f_1 \neq 0$, $|f_1| \neq 1$: formal series exists and converges; continuous iterates are analytic at 0
- case $|f_1| = 1$:
  - *f_1 a root of unity: no formal solution*
  - *f_1 not a root of unity: convergence depends on arithmetic conditions; e.g. Siegel $\log|f_1^n - 1| = \mathcal{O}(\log(x))$ as $n \to \infty$ sufficient*
  - *Baker $f(z) = \exp(z) - 1$: $f^{<s>}$ exists formally but diverges unless $n \in \mathbb{Z}$*
  - *Szekeres: there exists exactly one $f^{<s>}$ of which the formal series is an asymptotic expansion as $z \to 0$*
  - *idea: use Abel equation as $\sigma(z) = \exp(\alpha(z))$, $\alpha(z) \sim -1/z$*
  - *led to work by Écalle (Borel summability), Milnor (precise formulation of linearizability) and others*

The Feigenbaum functional equation 1

- consider scaling in the sequence of period-doubling bifurcations of maps $x \mapsto \mu - x^n$: we have orbit scaling α and parameter scaling δ
- solve for f: $f(x) = \gamma^{-1} f \circ f(\gamma x)$, $\gamma \equiv \alpha^{-1} = f(1)$
- $f(b) = 0$, $f(x) \sim c(b-x)^n$ for x near b
- there exist such regular solutions for each $n > 1$



The Feigenbaum functional equation 2

- what about cases where $f \to 0$ faster than any power of x near b?
- Szekeres' idea [8]: convert to coupled system, one of which is a Schröder or Abel equation:
  - *regular case:*

$$\begin{aligned} f \circ h(x) &= \gamma^2 f(x) \\ h(x) &= f(\gamma f(x)) \end{aligned}$$

  - *singular case: set $A(x) \propto \log(f(b-x))$ - we get $A(x) = c_{-2}/x^2 + c_{-1}/x + c_0 \log x + c + c_1 x + c_2 x^2 + \cdots$*
- the singular series are divergent but Borel summable: we get $\gamma_\infty = 0.0333810598\ldots$ and $\delta_\infty = 29.576303\ldots$
- Briggs & Dixon also solved the circle map case [8,9]
- $g \circ g(\epsilon^2 x) = \epsilon g(x), \quad \epsilon = g(1)$, obtaining $\epsilon_\infty = -0.275026971\ldots$

Abel's equation and regular growth

- we work here with real functions on $[0, \infty)$
- Abel: $\alpha \circ f(x) = \alpha(x) + 1$
- Abel: if α and α_1 are strictly increasing C^1 solutions, then $\alpha(x) - \alpha_1(x) = \psi \circ \alpha(x)$, where ψ is 1-periodic
- Lévy: which is the 'best' solution?
- let $c \geqslant 1$, $\mathcal{C}_c$ be the set of strictly convex analytic functions with $f(0) = 0$, $f'(0) = c$, $f''(x) > 0$, $\mathcal{C} = \cup_c \mathcal{C}_c$
- principal Abel function (best behaviour at 0): for $f(x) = \sum_{i=1}^{\infty} f_i\, x^i \in \mathcal{C}_c$:
  - $\alpha(x) = \log_c(x) + \mathcal{O}(x)$ *if* $f_1 = c > 1$
  - $\alpha(x) = -\frac{1}{f_2 x} \log(x) + \mathcal{O}(x)$ *if* $f_1 = c = 1$
- we have thus selected a solution by its behaviour at 0. Szekeres wants to study the behavior at ∞

More functional equations

- $D(x) \equiv 1/\alpha'(x)$ satisfies $D \circ f(x) = f'(x) D(x)$ (Julia)
- $t(x) \equiv \sum_{k=0}^{\infty} 1/f^{<k>\prime}(x)$
- t satisfies $t \circ f(x) = f'(x)(t(x) - 1)$
- if $f \in \mathcal{C}_c, c > 1$
  - $\alpha(x) = \log_c(x) + \mathcal{O}(x)$
  - $D(x) = \log(c)\left(x + \frac{f_2}{c(c-1)} x^2 + \ldots\right)$
  - $E \circ f(x) = f'(x)(E(x) + D'(x))$
  - $\phi(x) = \frac{1}{D(x)}\left((t(x) - 1) D'(x) - t'(x) D(x) + E(x)\right)$
- if $f \in \mathcal{C}_1$
  - $\alpha(x) = -1/(f_2 x) + \mathcal{O}(\log(x))$
  - $D(x) = f_2 x^2 + (f_3 - f_2^2) x^3 + \ldots$
  - $E \circ f(x) = f'(x) E(x) + D' \circ f(x)$
  - $\phi(x) = \frac{1}{D(x)}\left(t(x) D'(x) - t'(x) D(x) + E(x)\right)$

Yet more functional equations

- $\phi \circ f(x) = \phi(x) \quad \forall x \geqslant 0$
- $\psi(x) \equiv \phi(\alpha^{<-1>}(x))$ is 1-periodic, if α is the principal Abel function of f
- Szekeres' regularity criterion:
  - $\hat{\psi}_n \equiv \int_0^1 \exp(2\pi i n s)\, \psi(s) ds = \exp(\alpha_n + 2\pi i \beta_n)$
  - *this defines a mapping (LF sequence) $(\mathcal{C}) \to$ real-valued sequences α_n*
  - *defn: a sequence is completely monotonic if all differences of all orders are non-negative*
  - *defn: a sequence is L-regular if its first differences are the difference of two completely monotonic bounded sequences*
  - *equivalent to being a moment sequence: $\alpha_n = \int_0^1 t^n d\chi(t)$*
  - *defn: f is regularly growing if its LF sequence is L-regular*

Szekeres' experimental results

- $f(x) = 2x + x^2$: L-regular
- $f(x) = (1+x)^{e^2} - 1$: probably L-regular
- $f(x) = 3x + x^2$: not L-regular
- $f(x) = \exp(cx) - 1$: probably L-regular
- $f(x) = \exp(x) + x - 1$: not L-regular
- much more work needed to verify and extend these results!

Some speculation by KMB on Julia's equation

- $\iota \circ f = f'(x)\iota(x)$ for formal series $f(z) = \sum_{i=1}^{\infty} f_i\, z^i$
- if $f_2 \neq 0$, then $\iota = (f_3 - f_2^2)x^2 + \left(\frac{3}{2} f_2^3 + f_4 - \frac{5}{2} f_3 f_2\right) x^3 + \cdots$
- Julia inverse problem: any such series ι is formally conjugate to $\omega_{a,b} \equiv a x (x^k + b x^{2k})$ for appropriate a, b
- we can thus solve the ODE (for $k = 1$): $v'(x) = \frac{v^2(1-bv)}{x^2(1-bx)}$
- The exact solution of this is explicitly

$$b v(x)^{-1} = \begin{cases} 1 + W[+\exp\{\log(-b + \frac{1}{x}) + (\frac{1}{x} - c)/b - 1\}/b] & x < \frac{1}{b} \\ 1 & x = \frac{1}{b} \\ 1 + W[-\exp\{\log(+b - \frac{1}{x}) + (\frac{1}{x} - c)/b - 1\}/b] & x > \frac{1}{b} \end{cases}$$

where W is Lambert's W function ($W(z)\exp(W(z)) = z$)

More speculation by KMB on Julia's equation

- Jacobian conjecture: a polynomial map $f : \mathbb{R}^n \to \mathbb{R}^n$ with Jacobian determinant $\det J(f)$ equal to unity has a polynomial inverse
- we can write formally $f(z) = \exp(\omega D) z$, where D is the gradient operator and ω satisfies $J(f)\omega = \omega \circ f$ (multi-dimensional Julia equation)
- the mapping $f \mapsto \omega$ is a bijection (Labelle)
- then $f(z) = \exp(-\omega D) z$ ($z \in \mathbb{R}^n$)
- $\det J(f)$ can be expressed in terms of ω only
- we thus have a reformulation of the Jacobian conjecture: show that for all appropriate ω, both $\exp(\omega D) z$ and $\exp(-\omega D) z$ are polynomial

References

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