Statistics of continued fractions

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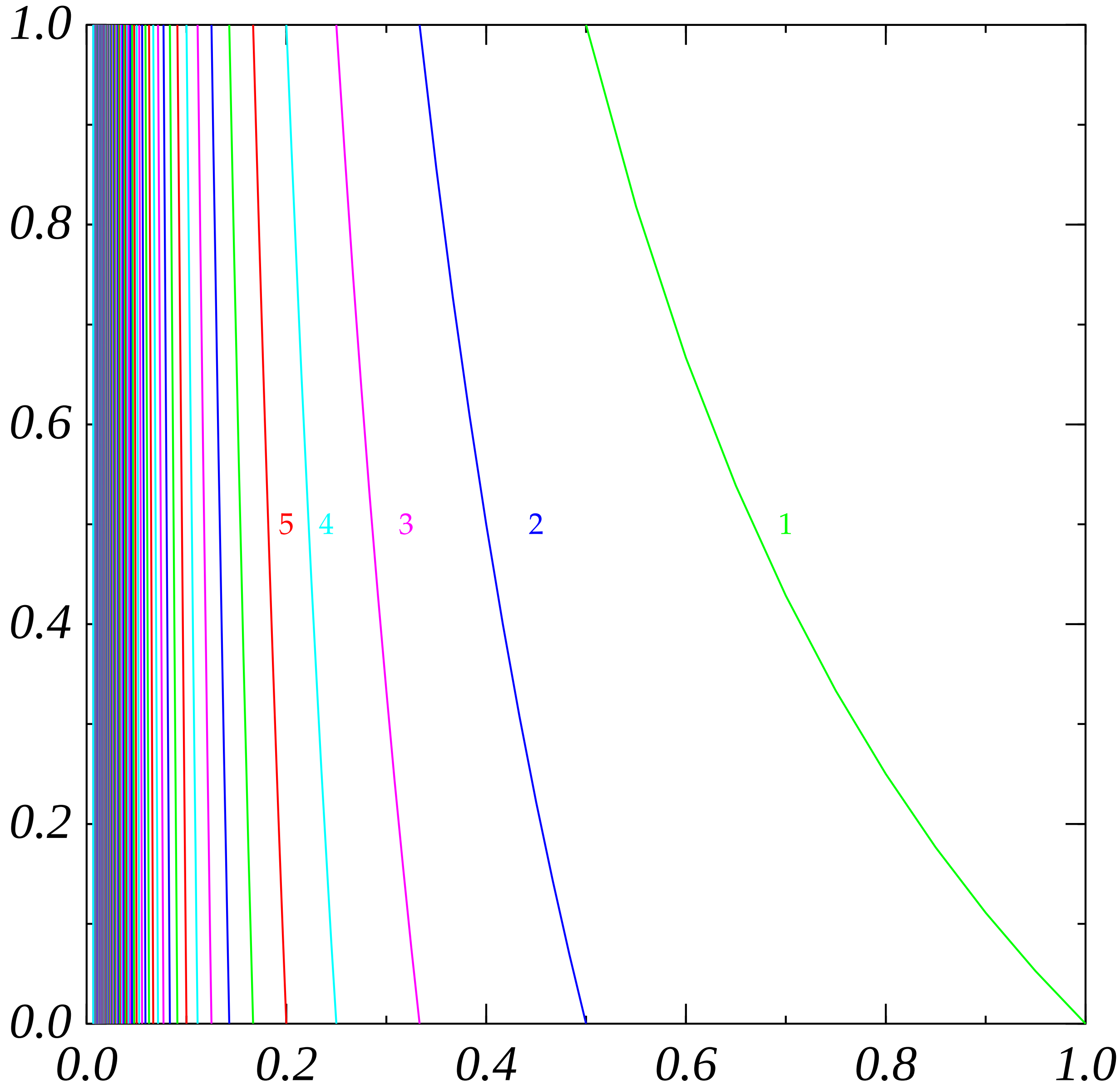
Classical theory

- Regular continued fractions are symbolic dynamics of the Gauss map:
$$g(x) = 1/x - \lfloor 1/x \rfloor \quad \text{for} \quad x \in (0, 1]$$
- The partial quotient ('digit') x_k output at the kth iteration is $x_k = \lfloor 1/x \rfloor$
- I write $x = [x_1, x_2, x_3, \ldots]$, where $x_k \in \{1, 2, 3, \ldots\}$
- The continued fraction is finite iff x is rational
- The continued fraction is eventually periodic iff x is a quadratic irrational
- For almost all x, the digit i occurs with relative frequency $\mu(i) \equiv \log_2 \left[\frac{(i+1)^2}{i(i+2)}\right]$

Gauss map



More theory

- I want to extend this theory to look at occurrences of finite blocks of digits $i=(i_1,i_2,\ldots,i_m), i_j \geqslant 1$
- [4, p226] gives a formula for relative frequency of the m-block i which holds as $n\to\infty$ for almost all irrationals:

$$\text{card}\{\kappa:(x_\kappa,\ldots,x_{\kappa+m-1})=i\ ,1\leqslant\kappa\leqslant n\}/n = \log_2\left[\frac{1+v(i)}{1+u(i)}\right]+o\left(n^{-1/2}\log^{(3+\epsilon)/2}(n)\right)$$

where (with $[i]=p_m/q_m$ for the m-block i)

$$u(i)=\begin{cases}(p_m+p_{m-1})/(q_m+q_{m-1}) & \text{if } m \text{ is odd}\\ p_m/q_m & \text{if } m \text{ is even}\end{cases}$$

$$v(i)=\begin{cases}p_m/q_m & \text{if } m \text{ is odd}\\ (p_m+p_{m-1})/(q_m+q_{m-1}) & \text{if } m \text{ is even}\end{cases}$$

Numerical values for the frequencies

For 2-blocks:

	1	2	3	4	5	6
1	0.15200	0.07038	0.04064	0.02647	0.01861	0.01380
2	0.07038	0.02914	0.01594	0.01005	0.00691	0.00505
3	0.04064	0.01594	0.00851	0.00529	0.00361	0.00262
4	0.02647	0.01005	0.00529	0.00326	0.00221	0.00160
5	0.01861	0.00691	0.00361	0.00221	0.00150	0.00108
6	0.01380	0.00505	0.00262	0.00160	0.00108	0.00078

Literature survey

- ★ Lang and Trotter [1] examined the frequency of digits amongst the first 1000 of several cubic irrationals
- ★ Brent et al. [2] examined the frequency of digits amongst the first 200000 of several algebraic irrationals
- ★ Neither of the above papers find any evidence of abnormality amongst the numbers examined
- ★ No papers look at the distribution of blocks of length greater than 1

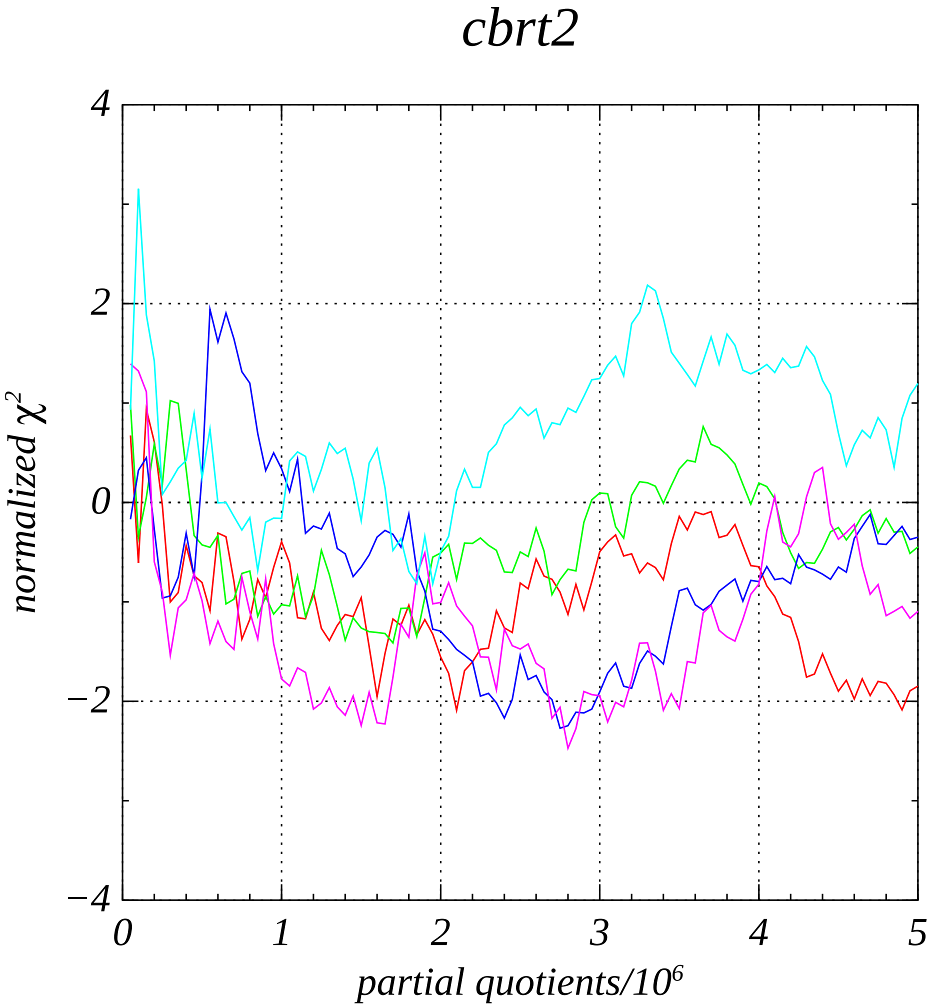
Explicit examples of abnormal numbers

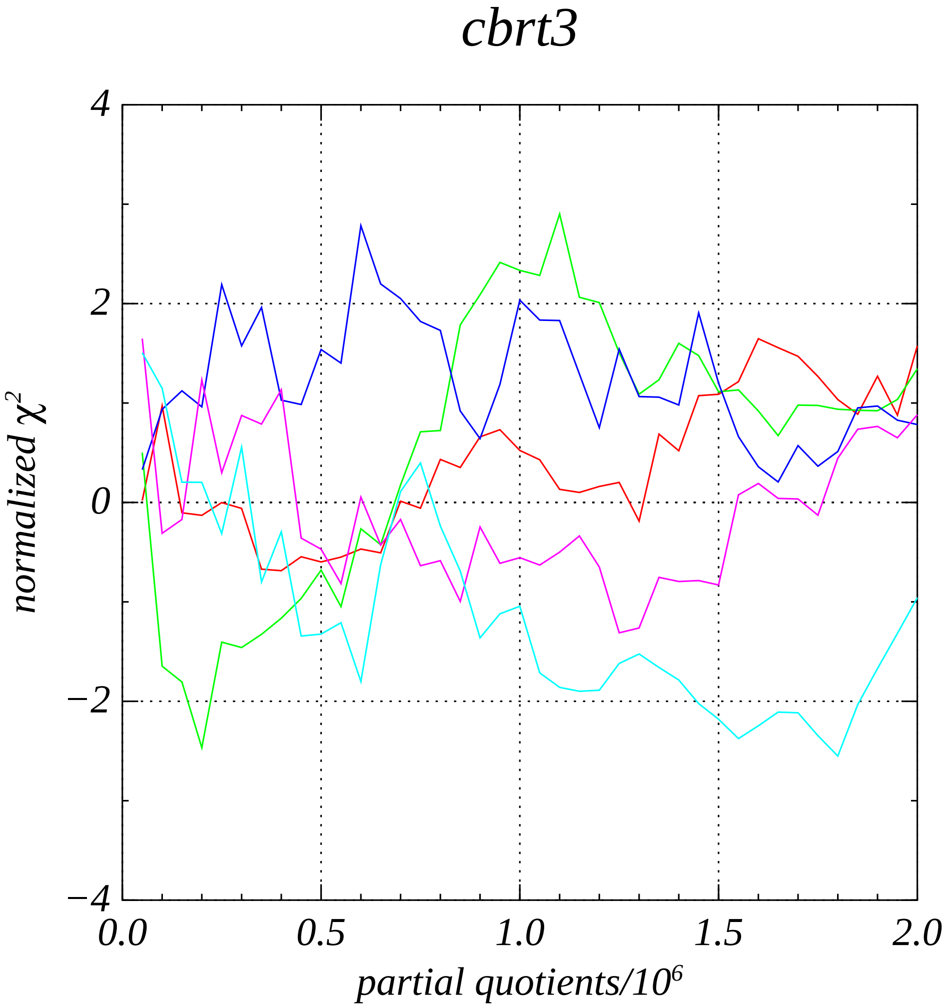
- ★ all quadratic irrationals, e.g. $2^{1/2} = 1+[2, 2, 2, 2, \ldots]$
- ★ $\mathsf{I}_1(2)/\,\mathsf{I}_0(2) = [1, 2, 3, 4, \ldots]$ (ratio of modified Bessel functions)
- ★ $\mathsf{I}_{1+a/d}(2/d)/\,\mathsf{I}_{a/d}(2/d) = [a+d, a+2d, a+3d, \ldots]$
- ★ $\tanh(1) = [1, 3, 5, 7, \ldots]$
- ★ $\exp(1/n) = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, \ldots];\ n = 1, 2, 3 \ldots$
- ★ $\exp(2) = 7+[2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, 1, 1, \ldots]$
- ★ $\exp(2/(2n+1));\ n = 1, 2, 3 \ldots$
- ★ $\sum_{k=1}^{\infty} 2^{-\lfloor k\phi \rfloor} = [2^0, 2^1, 2^1, 2^3, 2^5, 2^8, 2^{13}, \ldots];\ \phi = (\sqrt{5}-1)/2$

Method

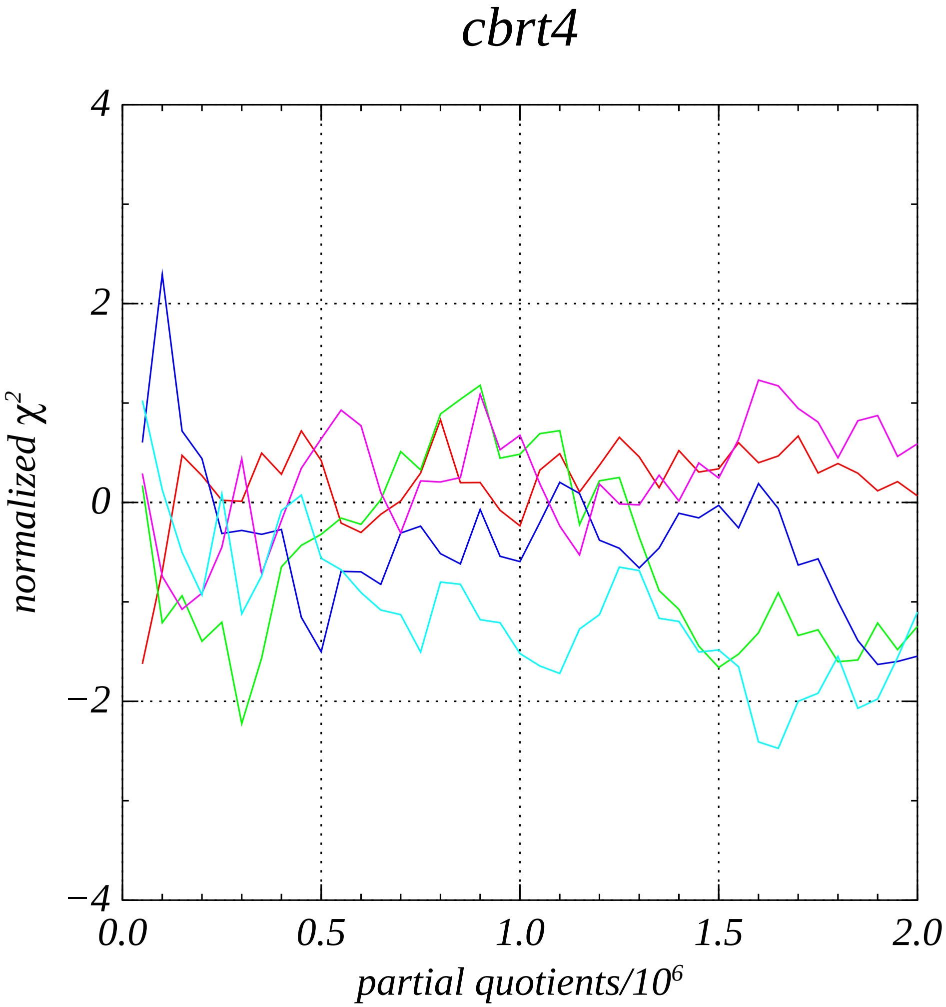
- I calculate a few million digits for several cubic irrationals and a few other irrationals
- I count exactly the observed frequency of all blocks of lengths 1,2,3,4, and 5
- I calculate a Pearson χ^2 test statistic which measures the deviation of the observed frequencies from the expected frequencies
- Because the number of degrees of freedom ν is so large (typically several thousand), a normal approximation is sufficiently accurate. The transformation is $Z \equiv \sqrt{2\chi^2} - \sqrt{2\nu - 1}$. Under the assumption of normality (of the cf of $x!$), Z is distributed $N(0,1)$

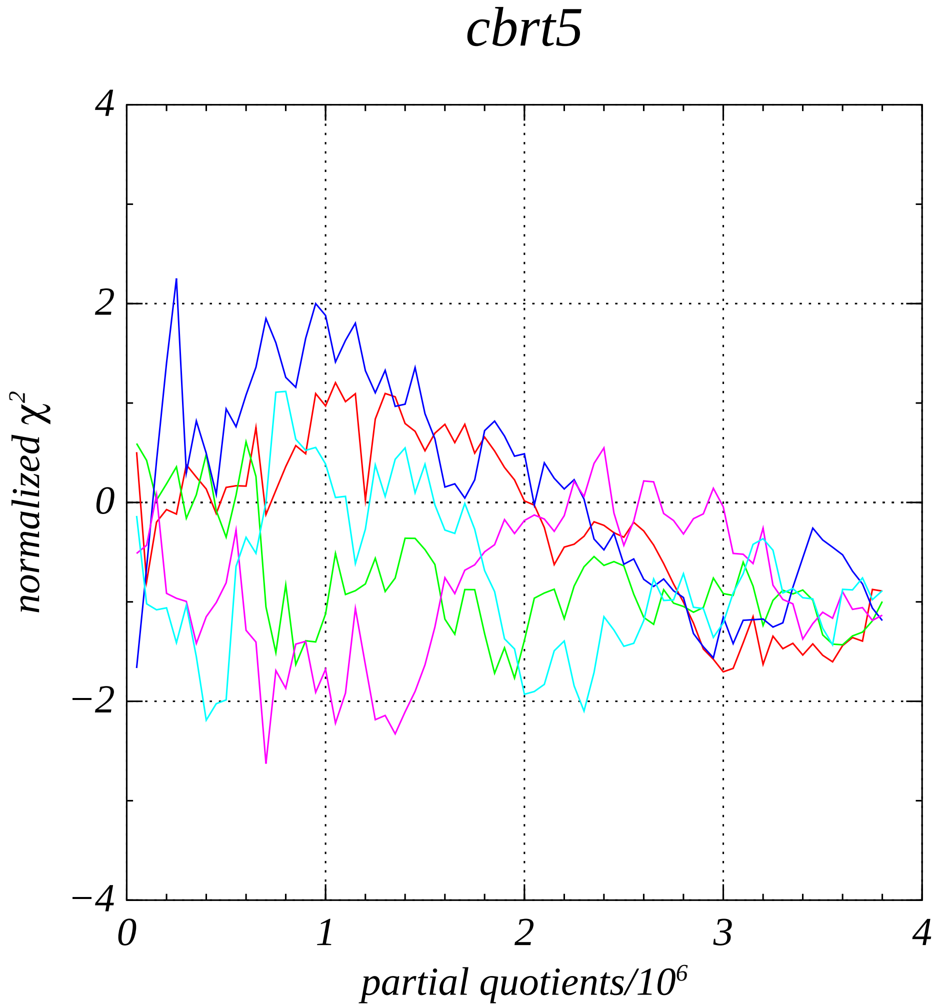
Pearson χ^2 results: $2^{1/3}$ and $3^{1/3}$



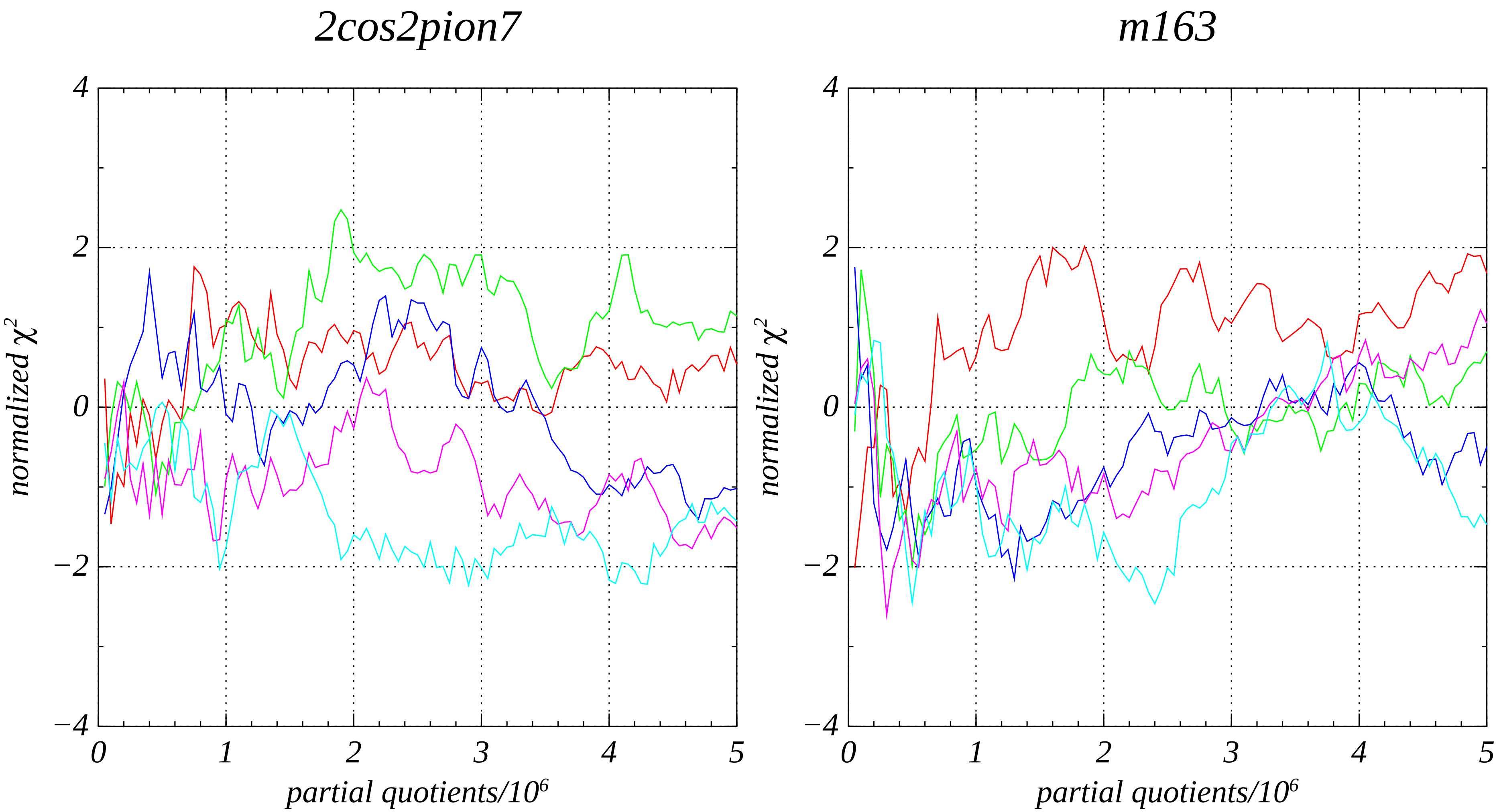


Pearson χ^2 **results:** $4^{1/3}$ **and** $5^{1/3}$



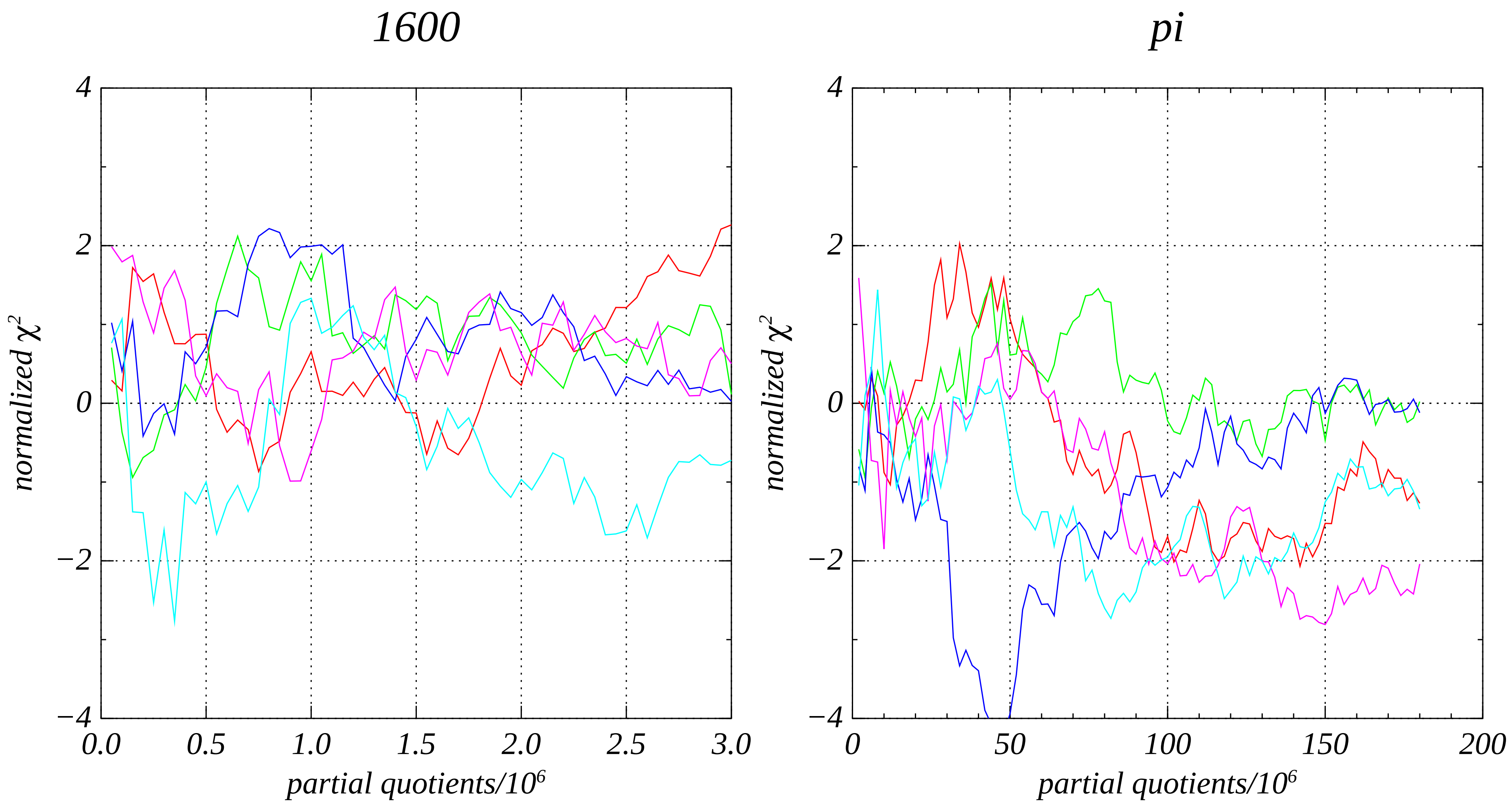


Pearson χ^2 results: $2\cos(2\pi/7)$ and largest root of $x^3-8x-10$



(the last example is famous for having several abnormally large digits)

Pearson χ^2 results: $(\sqrt{5}-1)/2+\sqrt{2}-1$ and π



Autocorrelation of digits

- ★ We would expect the the autocorrelation function (acf) of any analytic function of the digits that has a finite mean (for example, the log or the reciprocal) would decay like q^k at lag k, where $q \approx -0.303663$ is Wirsing's constant
- ★ This is investigated in the following graphs. I plot $\log_{10}$ of the absolute value of the acf as a function of lag. The green line has the Wirsing slope
- ★ In Rockett & Szüsz [3], we have the result

$$\Pr[x_n = r \;\&\; x_{n+k} = s] = \Pr[x_n = r]\Pr[x_{n+k} = s]\left(1+O(q^k)\right)$$

This, however, is too weak to allow explicit statistical tests

acf estimation difficulties

- For the AR(1) process $x(t+1) = \alpha x(t) + \epsilon, |\alpha| < 1$, the exact acf at lag k is $\rho(k) = \alpha^k$

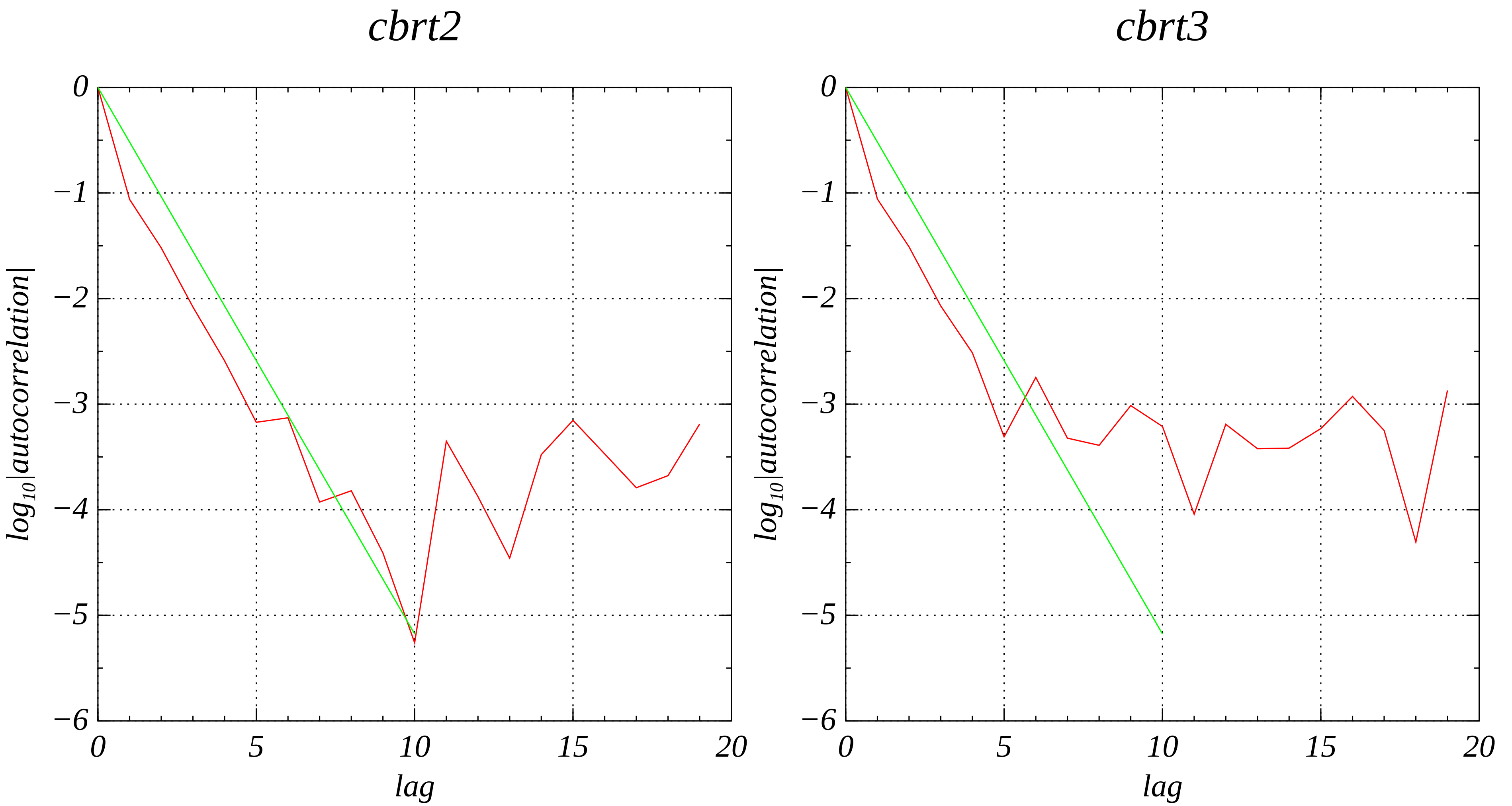
- But the usual acf estimator r for a sample of size n has variance

$$\text{var}\left[r_n(k)\right] = \frac{1}{n}\left[\frac{(1+\alpha^2)(1+\alpha^{2k})}{1-\alpha^2} - 2k\alpha^{2k}\right]$$

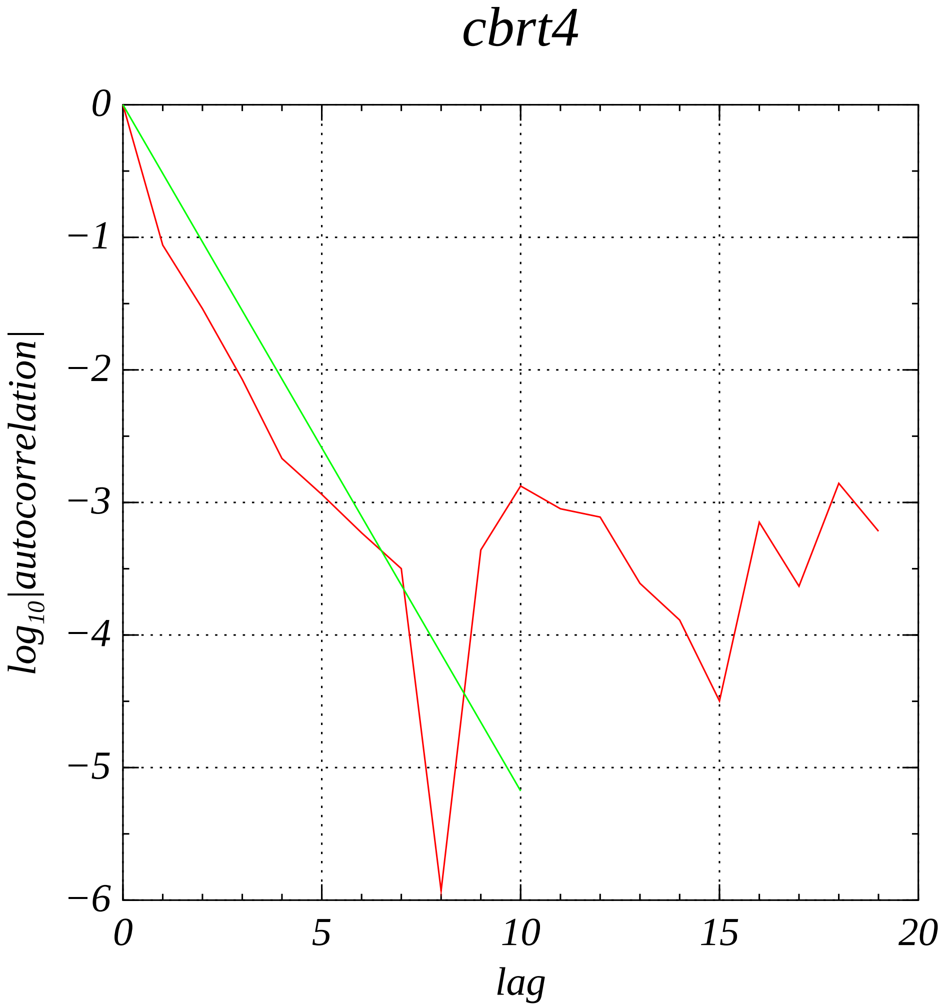
- More generally, for a process whose acf decays for large k in the same power-law fashion, we have approximate variance $\text{var}\left[r_n(k)\right] = \frac{1}{n}\left[\frac{1+\alpha^2}{1-\alpha^2}\right]$ for large k.

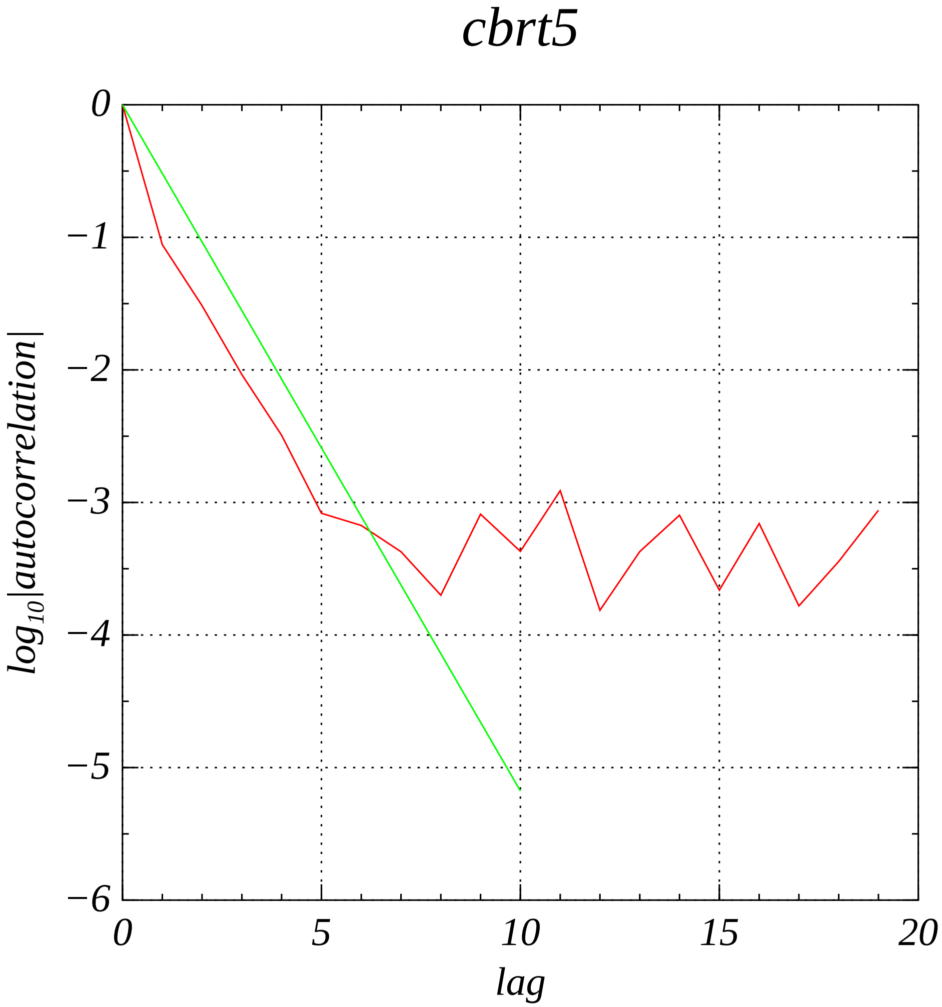
- I expect my process to conform to this behaviour, and if it does, putting in the numbers gives an estimate of $k = 6$ for the largest k for which the acf estimates are meaningful

autocorrelation of logs of digits: $2^{1/3}$ and $3^{1/3}$

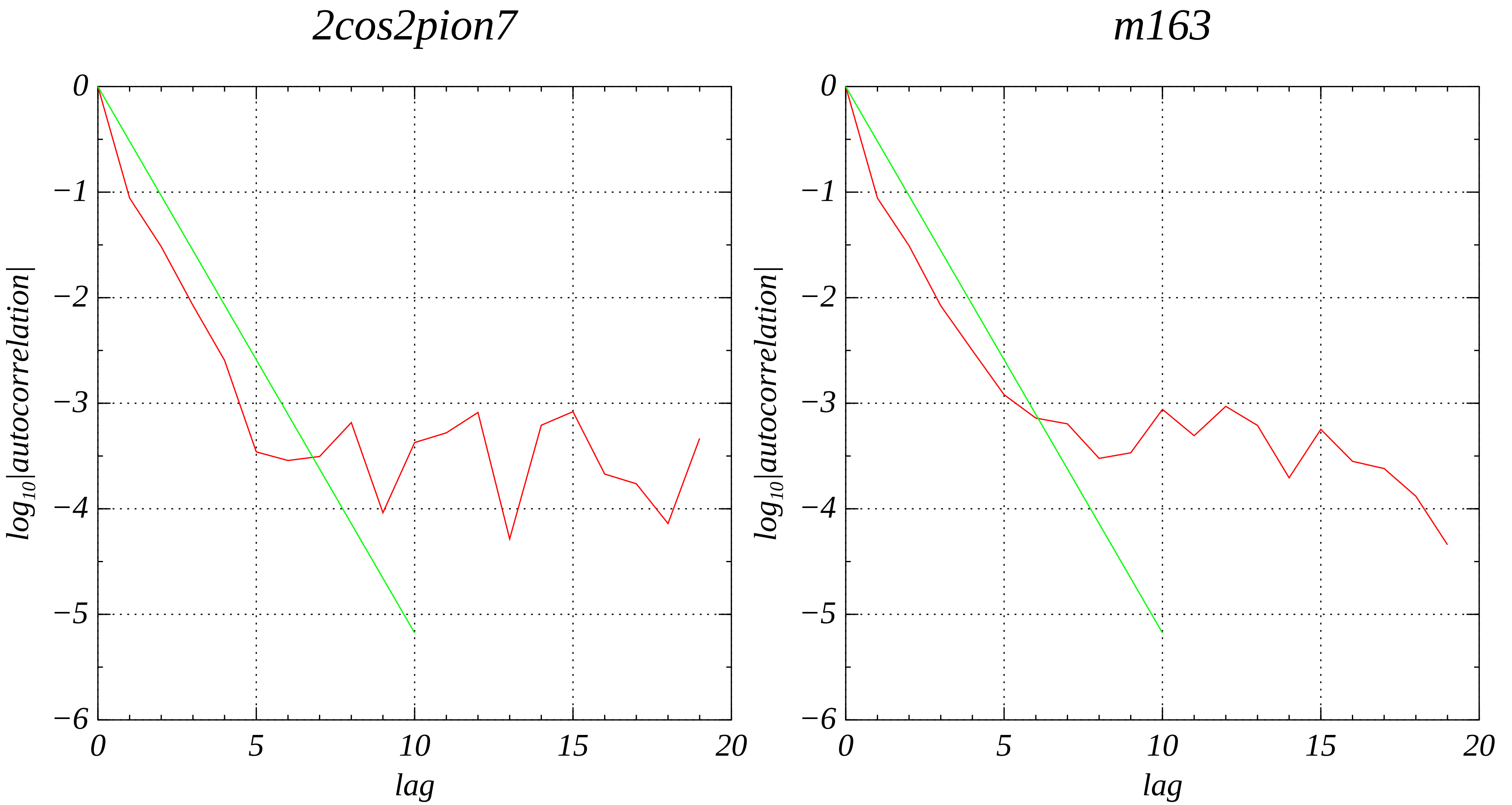


autocorrelation of logs of digits: $4^{1/3}$ and $5^{1/3}$





autocorrelation of logs of digits: $2\cos(2\pi/7)$ and largest root of $x^3 - 8x - 10$



References

[1] S Lang & H Trotter *Continued fractions for some algebraic numbers*, J. reine ang. Math. **255**, 112-134 (1972)

[2] R P Brent, A J van der Poorten & H J J te Riele *A comparative study of algorithms for computing continued fractions of algebraic numbers* in: Algorithmic Number Theory (H Cohen, ed.), LNCS, vol 1122, Springer-Verlag, 1996, 35-47

[3] A M Rockett & P Szüsz *Continued Fractions*, World Scientific 1992, ISBN 981-02-1052-3

[4] M Iosifescu & C Kraaikamp *Metrical Theory of Continued Fractions*, Kluwer 2002, ISBN 1402008929 (Mathematics and Its Applications, vol 547)