Continued fractions of algebraic numbers - some statistical tests

(preliminary report - work in progress)

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Abstract

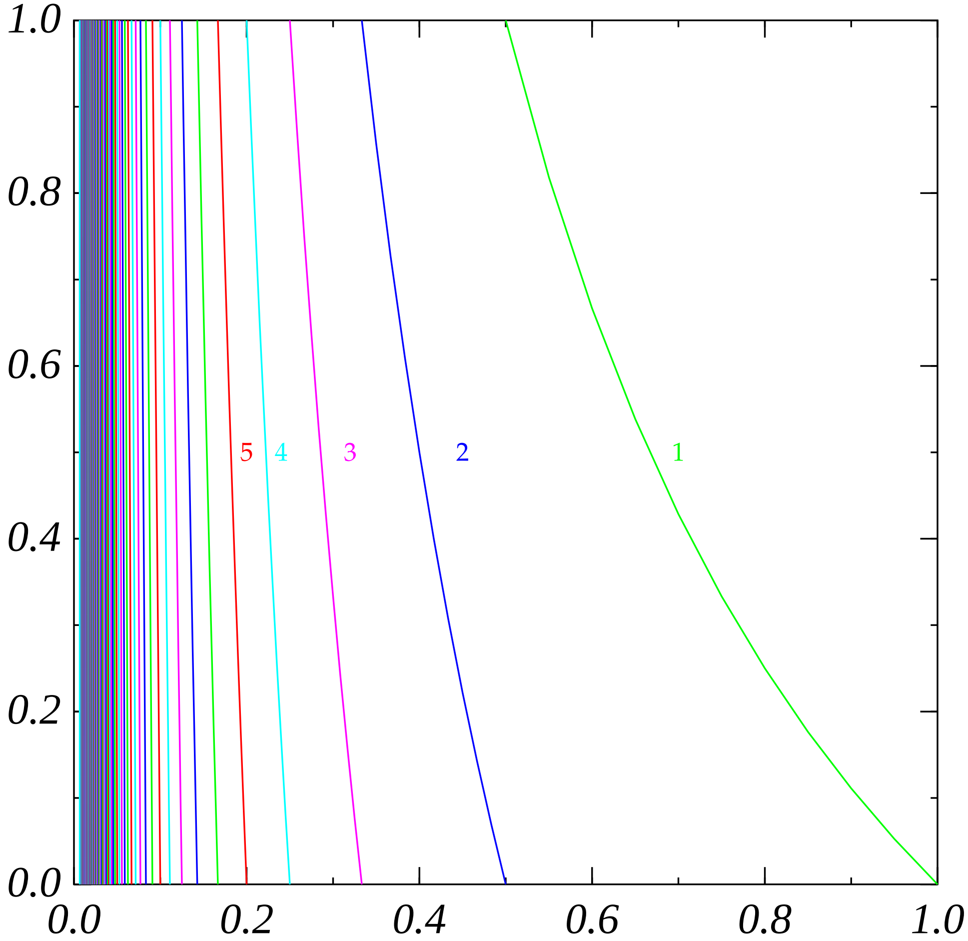
I study the frequency of occurence of blocks of partial quotients in the continued fractions of certain algebraic numbers (and π), with the aim of determining whether they conform to the expected frequencies known to hold for almost all irrational numbers.

Classical theory

- Regular continued fractions are symbolic dynamics of the Gauss map:
$$g(x) = 1/x - \lfloor 1/x \rfloor \quad \text{for} \quad x \in (0,1]$$
where the digit x_k (partial quotient) output at the kth iteration is $\lfloor 1/x \rfloor$
- We write $x = [x_1, x_2, x_3, \ldots]$, where $x_k \in \{1, 2, 3, \ldots\}$
- The continued fraction is finite iff x is rational
- For almost all x, the digit i occurs with relative frequency $\mu(i) \equiv \log_2 \left[\frac{(i+1)^2}{i(i+2)} \right]$
- The continued fraction is eventually periodic iff x is a quadratic irrational

Gauss map

$$g(x) = 1/x - \lfloor 1/x \rfloor \quad \text{for} \quad x \in (0, 1]$$



More theory

- I want to extend this theory to look at occurrences of finite blocks of digits $i=(i_1,i_2,\ldots,i_m), i_j \geqslant 1$
- [8, p226] gives a formula for relative frequency of the m-block i which holds as $n\to\infty$ for almost all irrationals:

$$\text{card}\{\kappa : (x_\kappa,\ldots,x_{\kappa+m-1})=i\ , 1\leqslant\kappa\leqslant n\}/n = \log_2\left[\frac{1+v(i)}{1+u(i)}\right] + o\left(n^{-1/2}\log^{(3+\epsilon)/2}(n)\right)$$

where

$$u(i)=\begin{cases}\frac{p_m+p_{m-1}}{q_m+q_{m-1}} & \text{if } m \text{ is odd}\\ \frac{p_m}{q_m} & \text{if } m \text{ is even}\end{cases}$$

$$v(i)=\begin{cases}\frac{p_m}{q_m} & \text{if } m \text{ is odd}\\ \frac{p_m+p_{m-1}}{q_m+q_{m-1}} & \text{if } m \text{ is even}\end{cases}$$

with $[i]=p_m/q_m$ for the m-block i.

★ For blocks of length 2 and 3, we have explicitly:

$$\mu(i)=\log_2\left[\frac{(1+i_1\,i_2)\;(2+i_1+i_2+i_1\,i_2)}{(1+i_1+i_1\,i_2)\;(1+i_2+i_1\,i_2)}\right]$$

and

$$\mu(i)=\log_2\left[\frac{(1+i_1+i_3+i_2\,i_3+i_1\,i_2\,i_3)\;(1+i_1+i_3+i_1\,i_2+i_1\,i_2\,i_3)}{(i_1+i_3+i_1\,i_2\,i_3)\;(2+i_1+i_2+i_3+i_2\,i_3+i_1\,i_2+i_1\,i_2\,i_3)}\right]$$

★ For length 4,

$$\mu(i)=\log_2\left[\frac{(2+i_3+i_4+i_1+i_2\;(1+i_3\;(1+i_4))\;(1+i_1)+i_4\;(i_3+i_1))\;(1+(i_2+i_4)\;i_1+i_3\,i_4\;(1+i_2\,i_1))}{(1+i_2\;(1+i_3\,i_4)\;(1+i_1)+i_4\;(1+i_3+i_1))\;(1+(1+i_2+i_4)\;i_1+i_3\;(1+i_4)\;(1+i_2\,i_1))}\right]$$

Numerical values for the frequencies

For 2-blocks:

	1	2	3	4	5	6
1	0.15200	0.07038	0.04064	0.02647	0.01861	0.01380
2	0.07038	0.02914	0.01594	0.01005	0.00691	0.00505
3	0.04064	0.01594	0.00851	0.00529	0.00361	0.00262
4	0.02647	0.01005	0.00529	0.00326	0.00221	0.00160
5	0.01861	0.00691	0.00361	0.00221	0.00150	0.00108
6	0.01380	0.00505	0.00262	0.00160	0.00108	0.00078

More theory

- Note that $\mu(i)$ is unchanged if we reverse the block i, whatever the length. I do not know what other symmetries exist
- If a particular x has all blocks occurring with these expected frequencies, we call x *normal*
- Note that because of the rapid decay of correlations (approximately $(-0.3)^n$ at lag n), there is not much point in studying very long blocks ($n > 5$, say). For long blocks, the two ends are effectively independent. This makes an empirical study such as the present one feasible.
- Of course, we can never *prove* abnormality (if it exists) merely by a statistical analysis of a finite portion of the infinite continued fraction. However, we might hope to find *evidence* of abnormality, which can then be proven by other methods

Literature survey

- ★ [1] examines the first 200 digits of the real root of $x^3-8x-10$; in particular it explains the occurrence of several very large digits
- ★ [2] examines the frequency of digits amongst the first 1000 of several cubic irrationals
- ★ [3] examines the frequency of digits amongst the first 200000 of several algebraic irrationals
- ★ None of the above papers find any evidence of abnormality amongst the numbers examined
- ★ In [7], we have the result $\Pr\left[x_n=r \,\&\, x_{n+k}=s\right] = \Pr\left[x_n=r\right]\Pr\left[x_{n+k}=s\right]\left(1+O(q^k)\right)$, where $q\approx-0.303663$ is the Gauss-Kuzmin-Wirsing constant. This, however, is too weak a result to allow explicit statistical tests
- ★ No papers look at the distribution of blocks of length greater than 1

Explicit examples of abnormal numbers

- all quadratic irrationals, e.g. $2^{1/2} = 1+[2, 2, 2, 2, \ldots]$
- $\mathsf{I}_1(2)/\,\mathsf{I}_0(2) = [1, 2, 3, 4, \ldots]$ (ratio of modified Bessel functions)
- $\mathsf{I}_{1+a/d}(2/d)/\,\mathsf{I}_{a/d}(2/d) = [a+d, a+2d, a+3d, \ldots]$
- $\tanh(1) = [1, 3, 5, 7, \ldots]$
- $\exp(1/n) = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, \ldots];\ n = 1, 2, 3\ldots$
- $\exp(2) = 7+[2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, 1, 1, 12, 54, \ldots]$
- $\exp(2/(2n+1));\ n = 1, 2, 3\ldots$
- $\sum_{k=1}^{\infty} 2^{-\lfloor k\phi \rfloor} = [2^0, 2^1, 2^1, 2^3, 2^5, 2^8, 2^{13}, \ldots];\ \phi = (\sqrt{5}-1)/2$

Other theory

- It is known that for almost all x, the mean of the digits does not exist. However, the mean of the log and mean of the reciprocal do exist and are approximately 0.98784905683381078769204 and 1.745405662407346 respectively. All my examples give results consistent with these.

- Similarly for the mean of $(x_j)^{-k}, k = 2, 3, 4, \ldots, 10$

Method

- ⋆ I calculate a few million digits for several cubic irrationals and a few other irrationals
- ⋆ I count exactly the observed frequency of all blocks of lengths 1,2,3,4,5. I use non-overlapping blocks (thanks to Gesine Reinert for this suggestion)
- ⋆ I calculate a Pearson χ^2 test statistic which measures the deviation of the observed frequencies from the expected frequencies
- ⋆ Because the number of degrees of freedom ν is so large (typically several thousand), a normal approximation is sufficiently accurate. The transformation is $Z \equiv \sqrt{2\chi^2} - \sqrt{2\nu - 1}$. Under the assumption of normality (of the cf of x!), Z is distributed $N(0,1)$

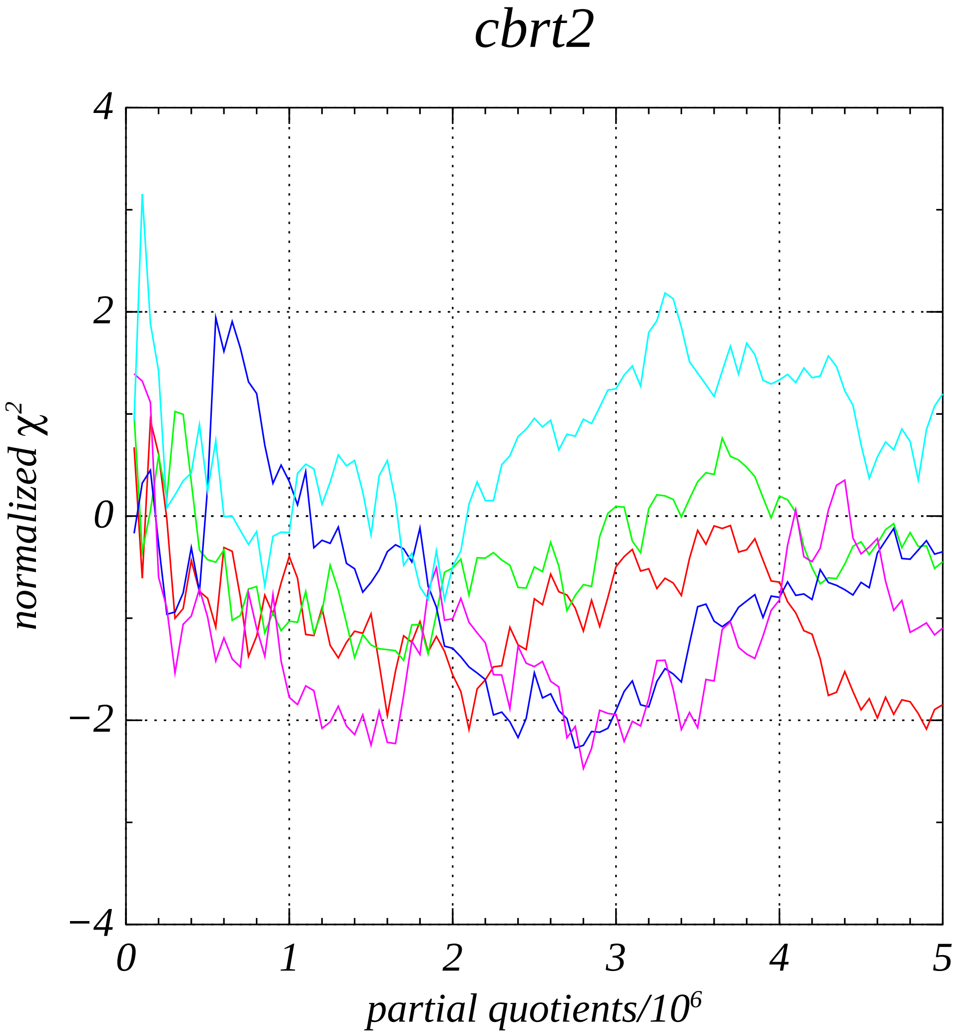
- I plot this Z for blocks of length 1 (red), 2 (green), 3 (dark blue), 4 (light blue), 5 (violet) as a function of the number of digits computed. We are looking for large deviations (say, >3) away from zero as a sign of abnormality

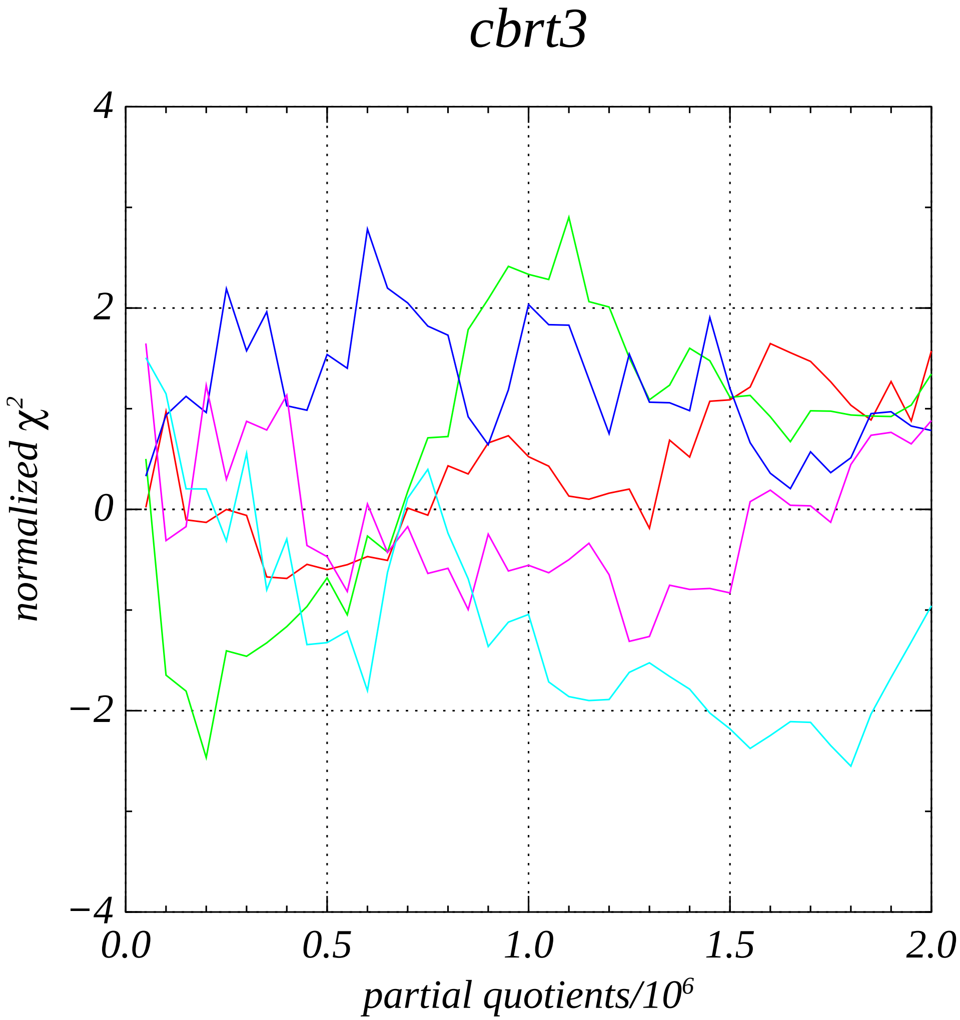
- The rare events have to handled with care. I used this prescription: for each event i, I calculate its expected frequency. If this is less than 5, I classify this as a rare event, and I lump together all rare events into one bin. If it is greater than 5, I calculate the usual contribution to χ^2 (i.e. $(\text{observed}-\text{expected})^2/\text{expected}$). There is also a tail correction term to χ^2, namely those events which were never observed to occur

- There are probably better ways of doing this!

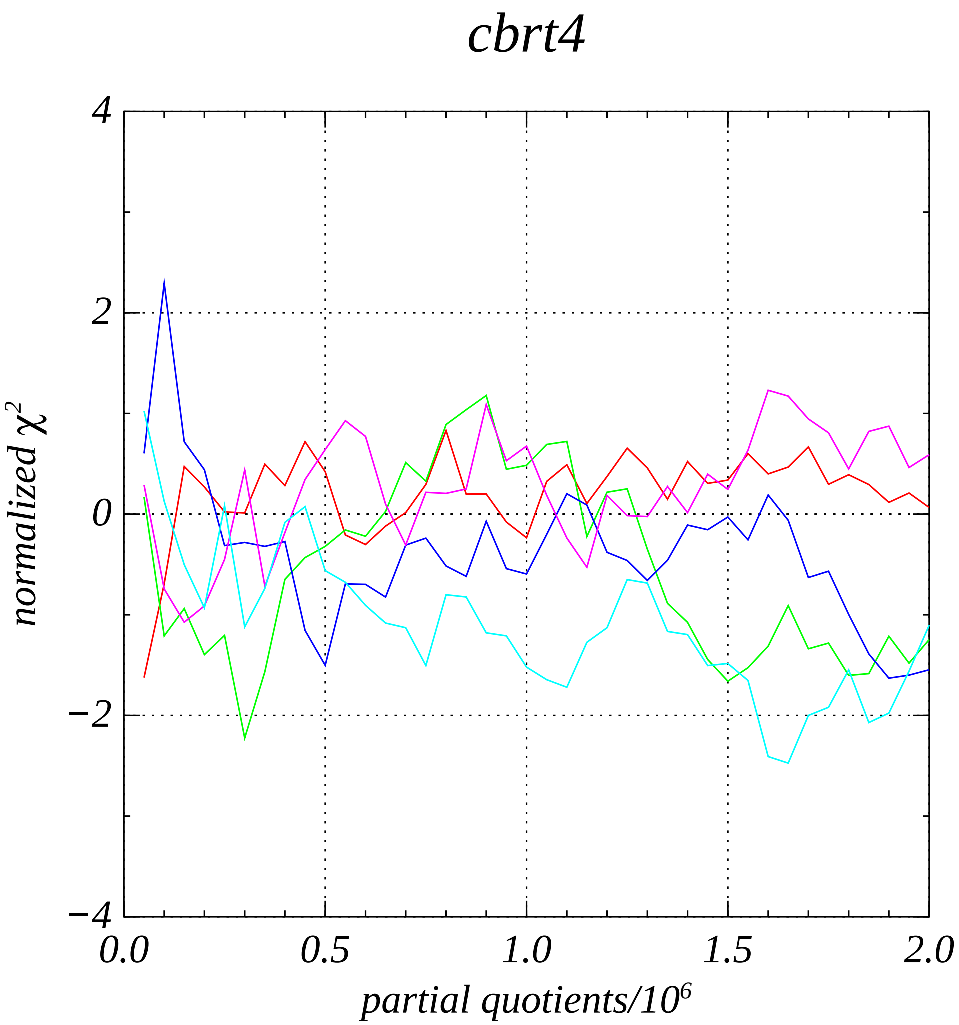
- I also do likelihood ratio tests, where the test statistic is $2\sum$ observed $\log\frac{\text{observed}}{\text{expected}}$ However, this is probably not very different, since when x (observed) is close to y (expected), $2\sum x\log(x/y)\approx\sum(x-y)^2/y$

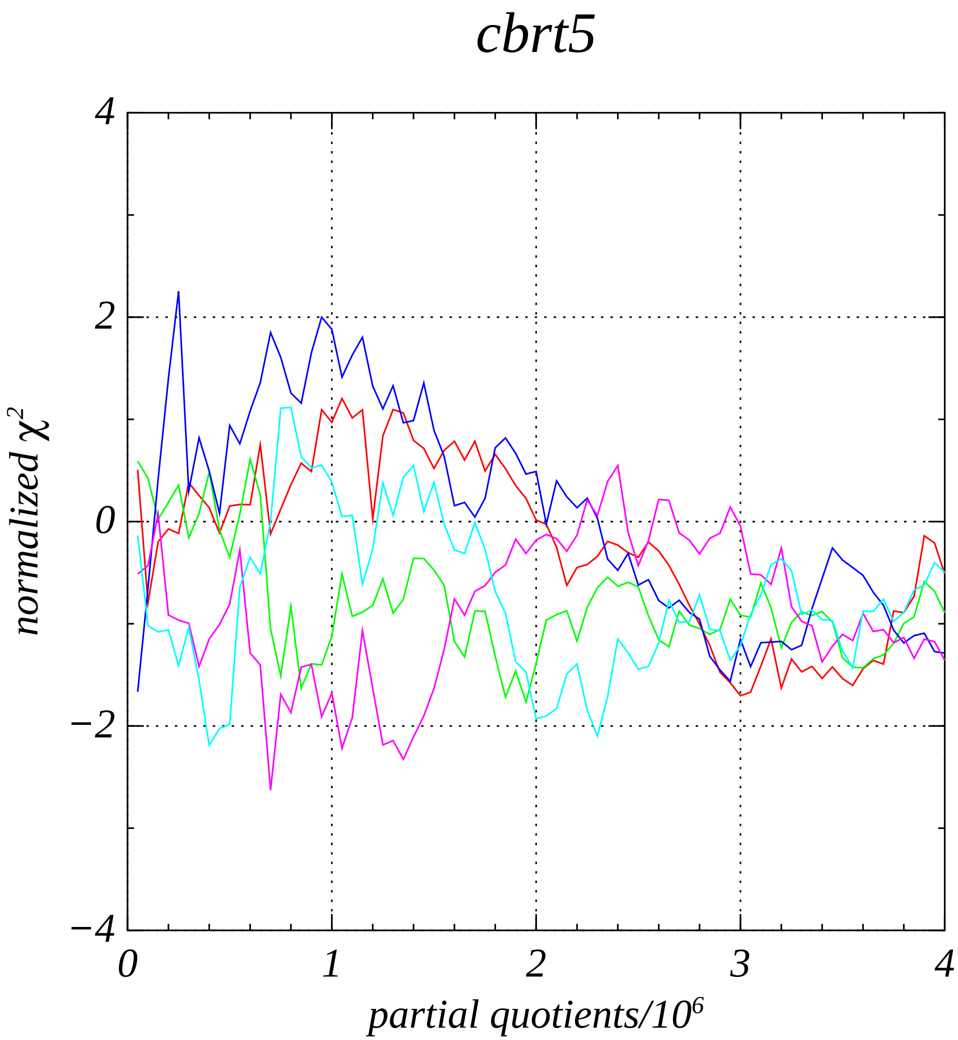
Pearson χ^2 results: $2^{1/3}$ and $3^{1/3}$



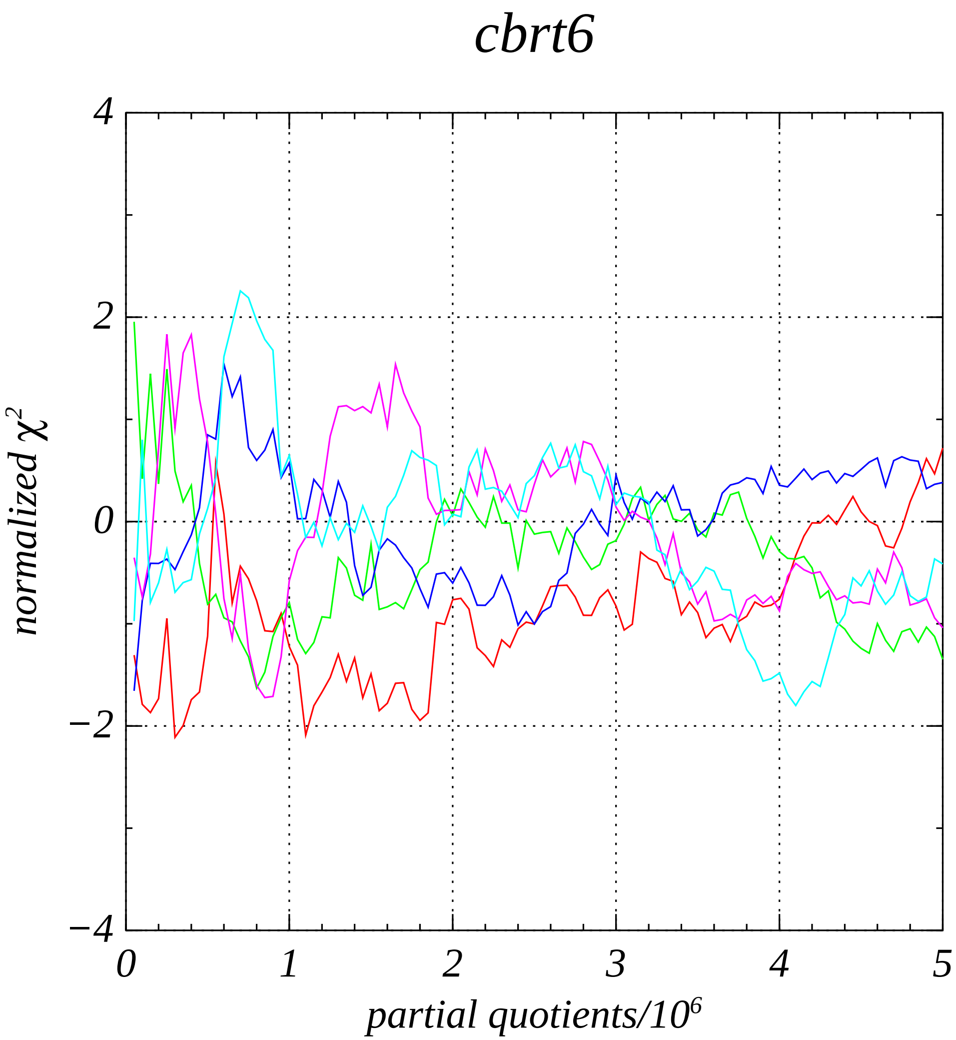


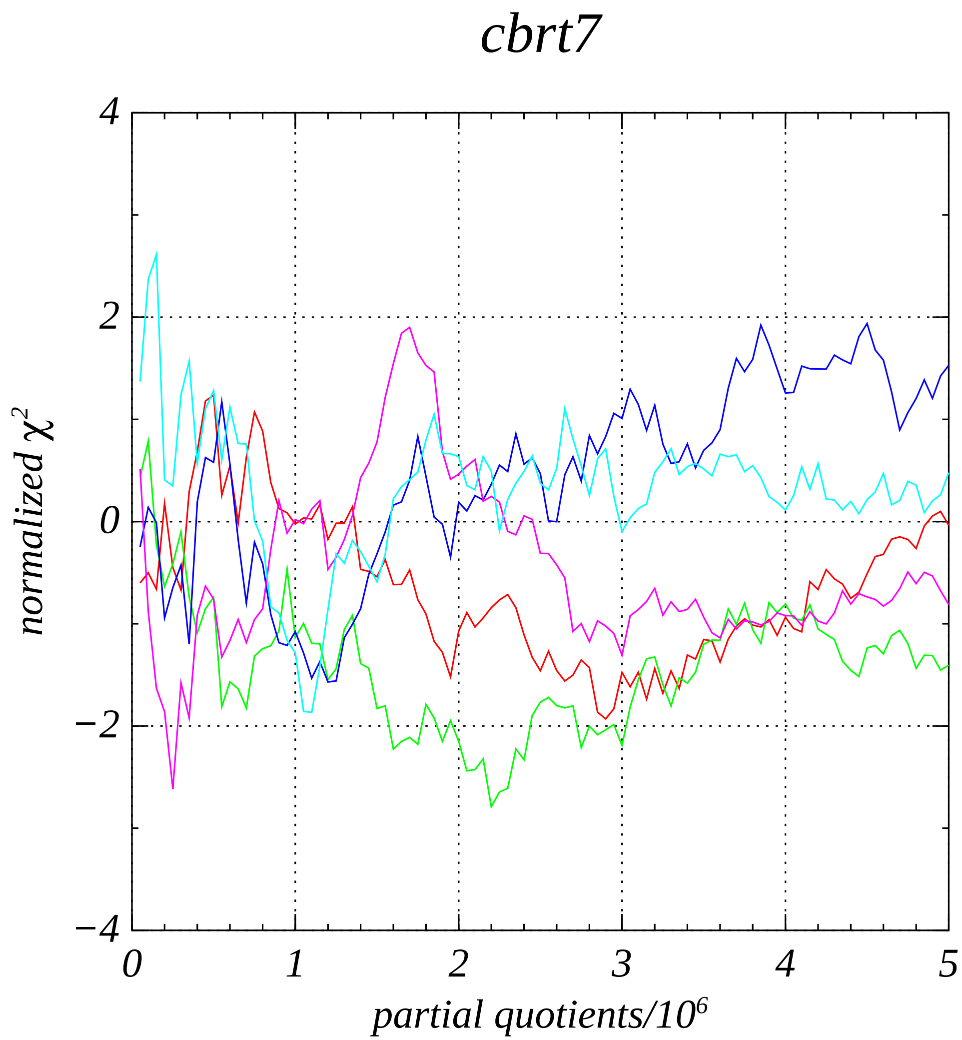
Pearson χ^2 results: $4^{1/3}$ and $5^{1/3}$



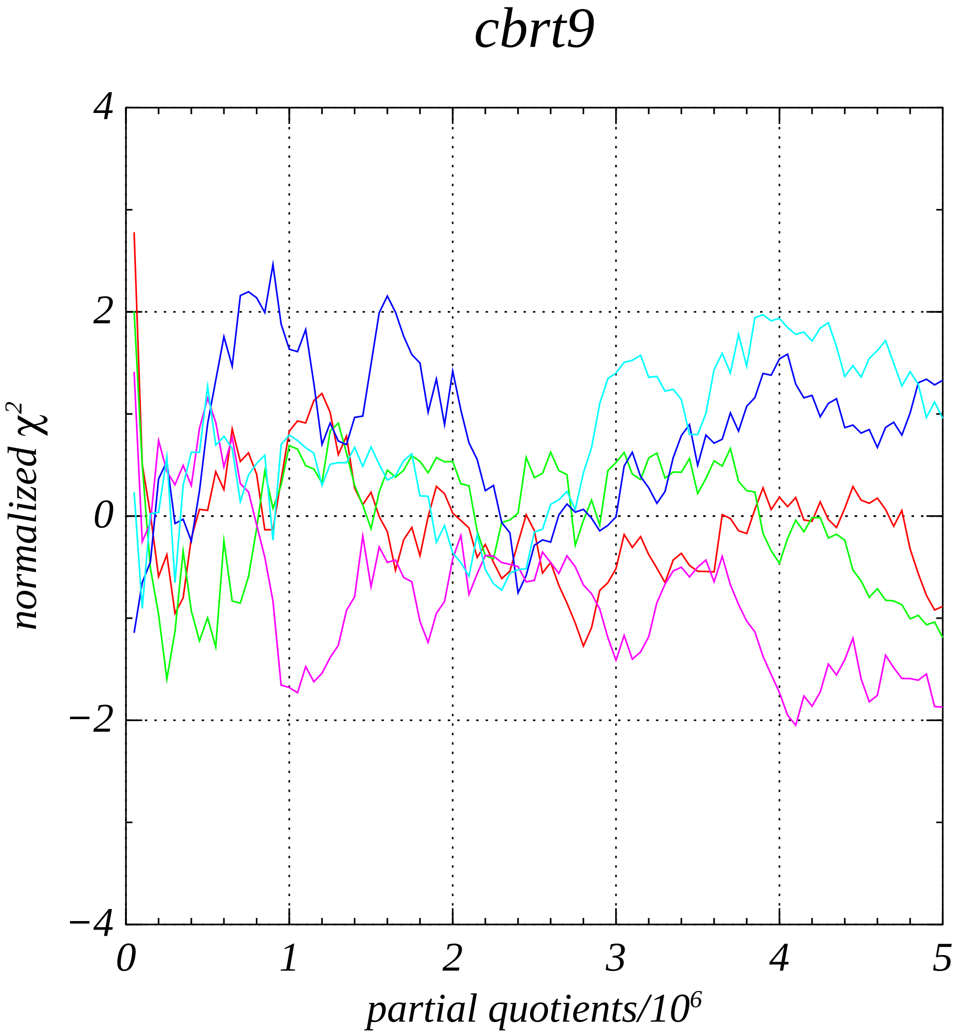


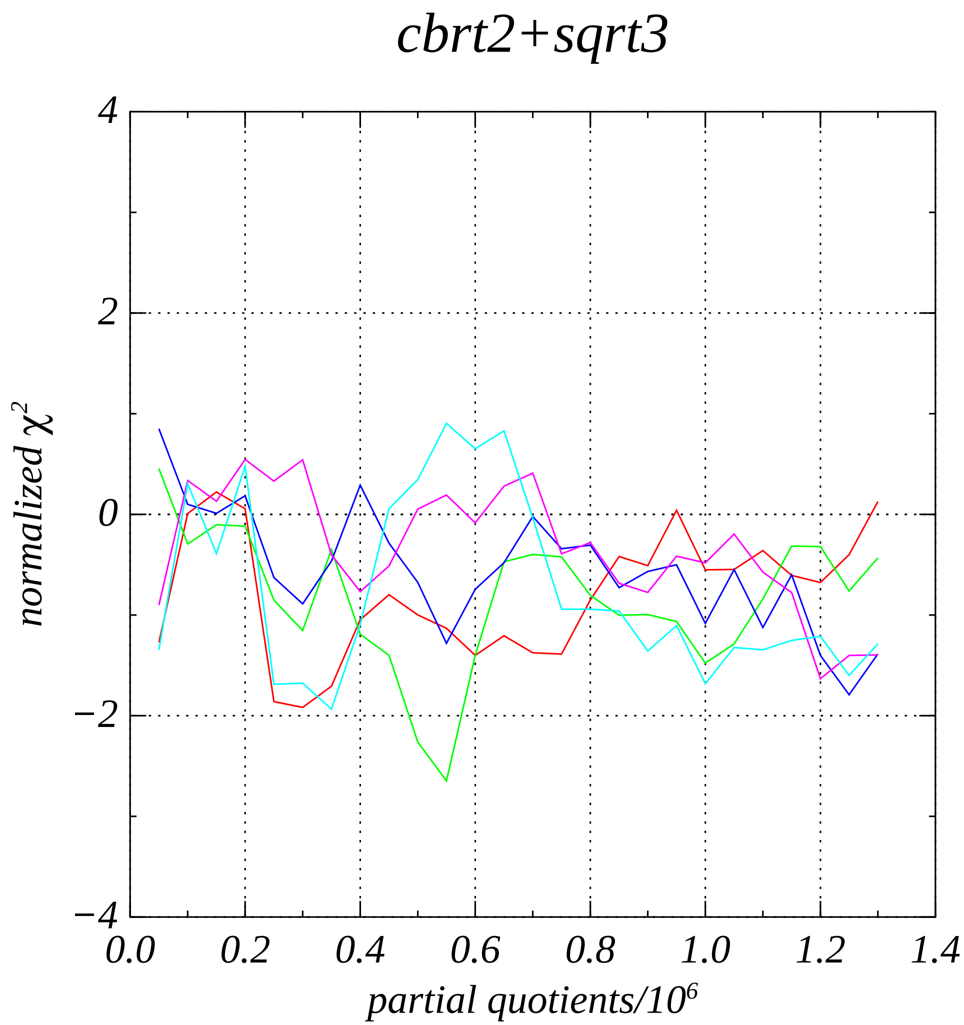
Pearson χ^2 results: $6^{1/3}$ and $7^{1/3}$



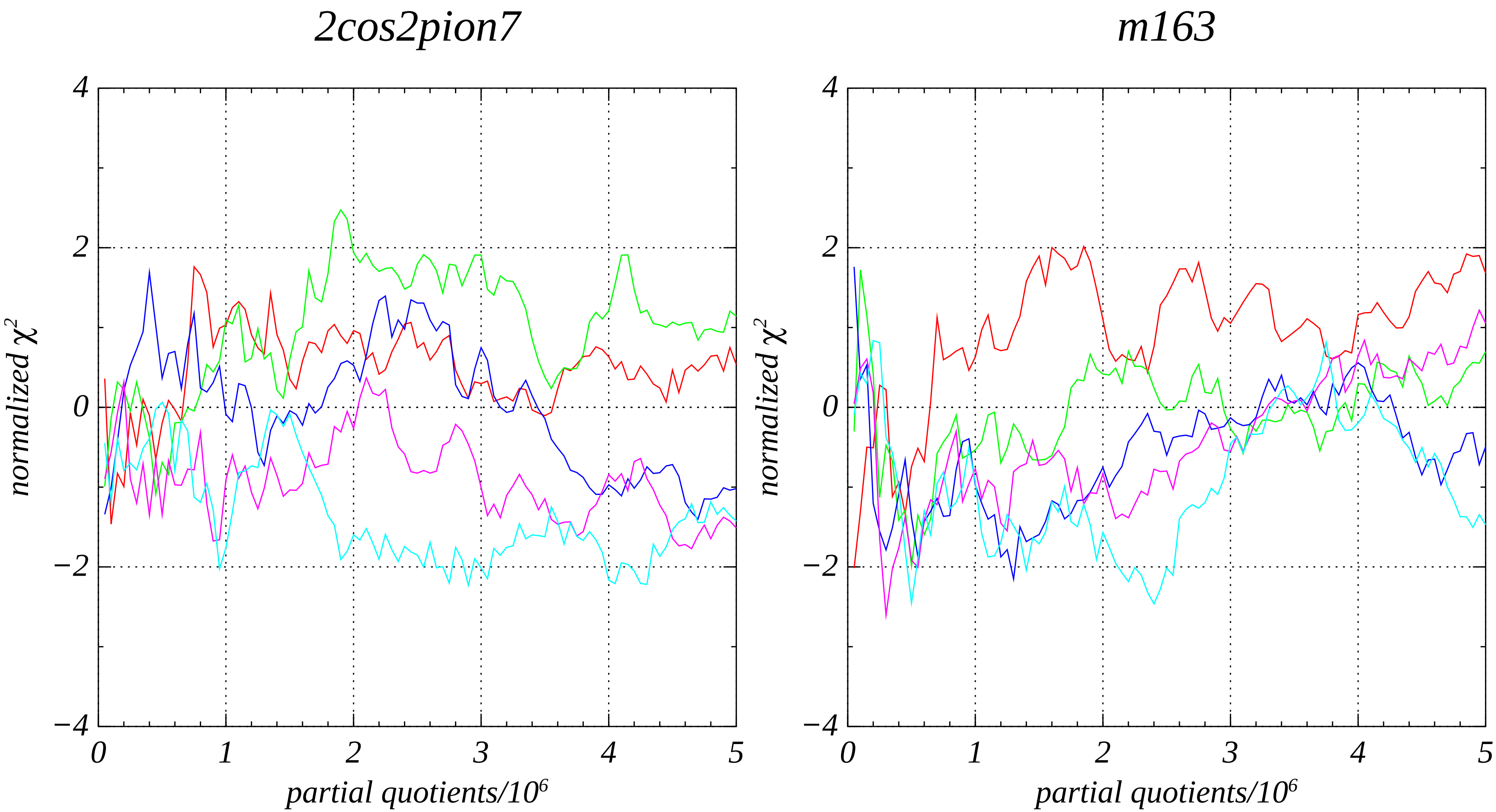


Pearson χ^2 results: $9^{1/3}$ and $2^{1/3}+3^{1/2}$



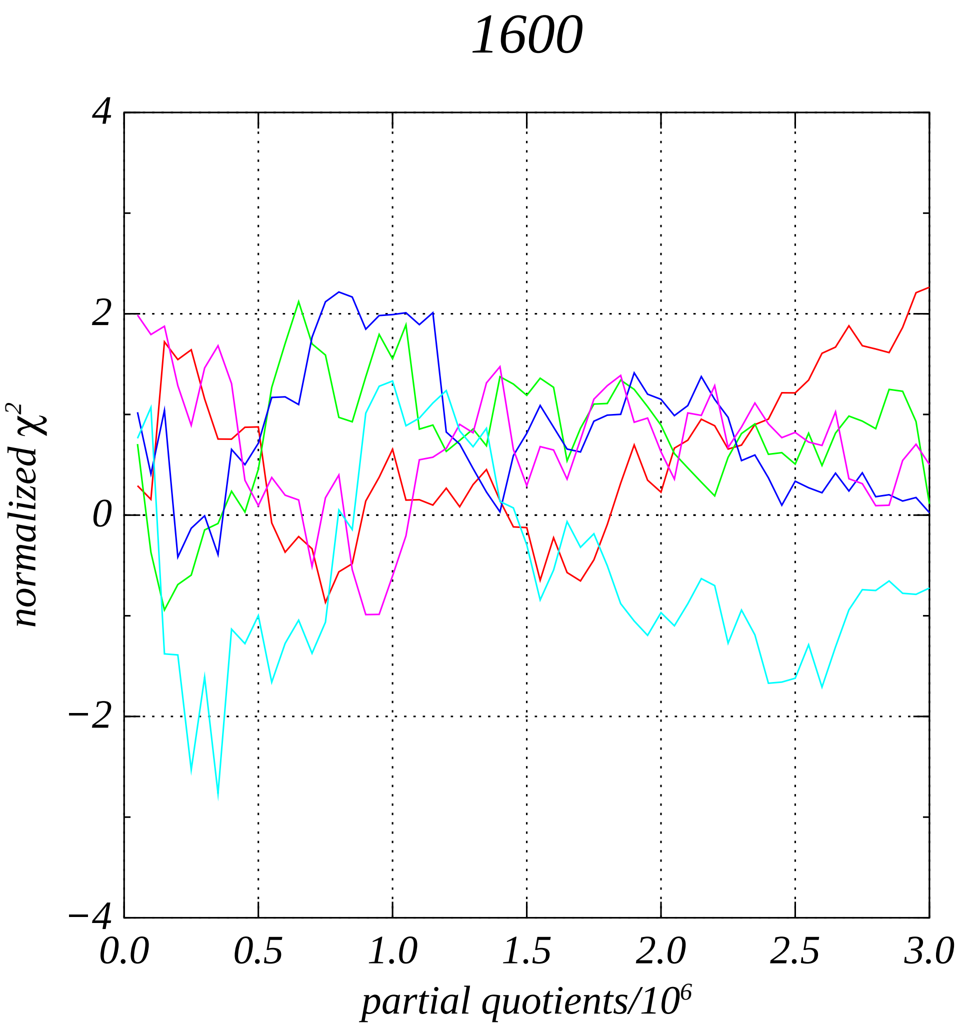


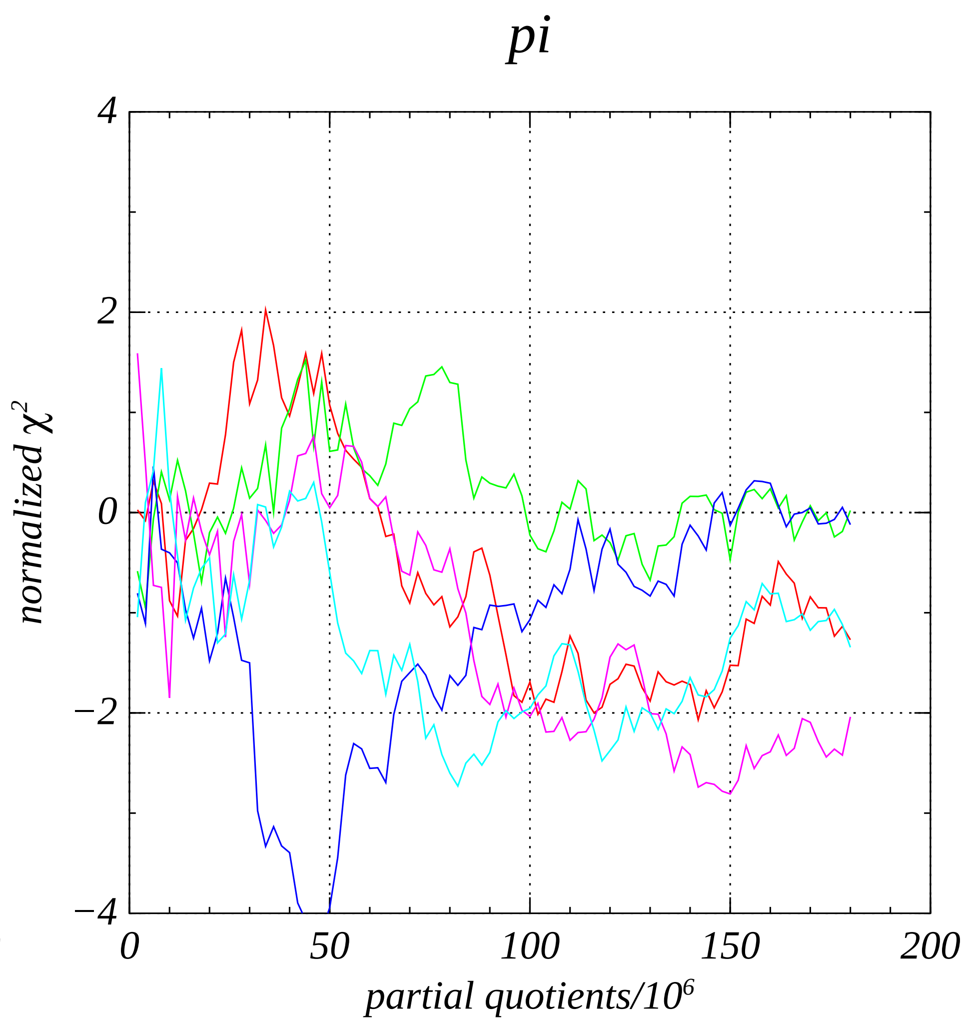
Pearson χ^2 results: $2\cos(2\pi/7)$ and largest root of $x^3-8x-10$



(the last example is famous for having several abnormally large digits)

Pearson χ^2 results: $(\sqrt{5}-1)/2+\sqrt{2}-1$ and π





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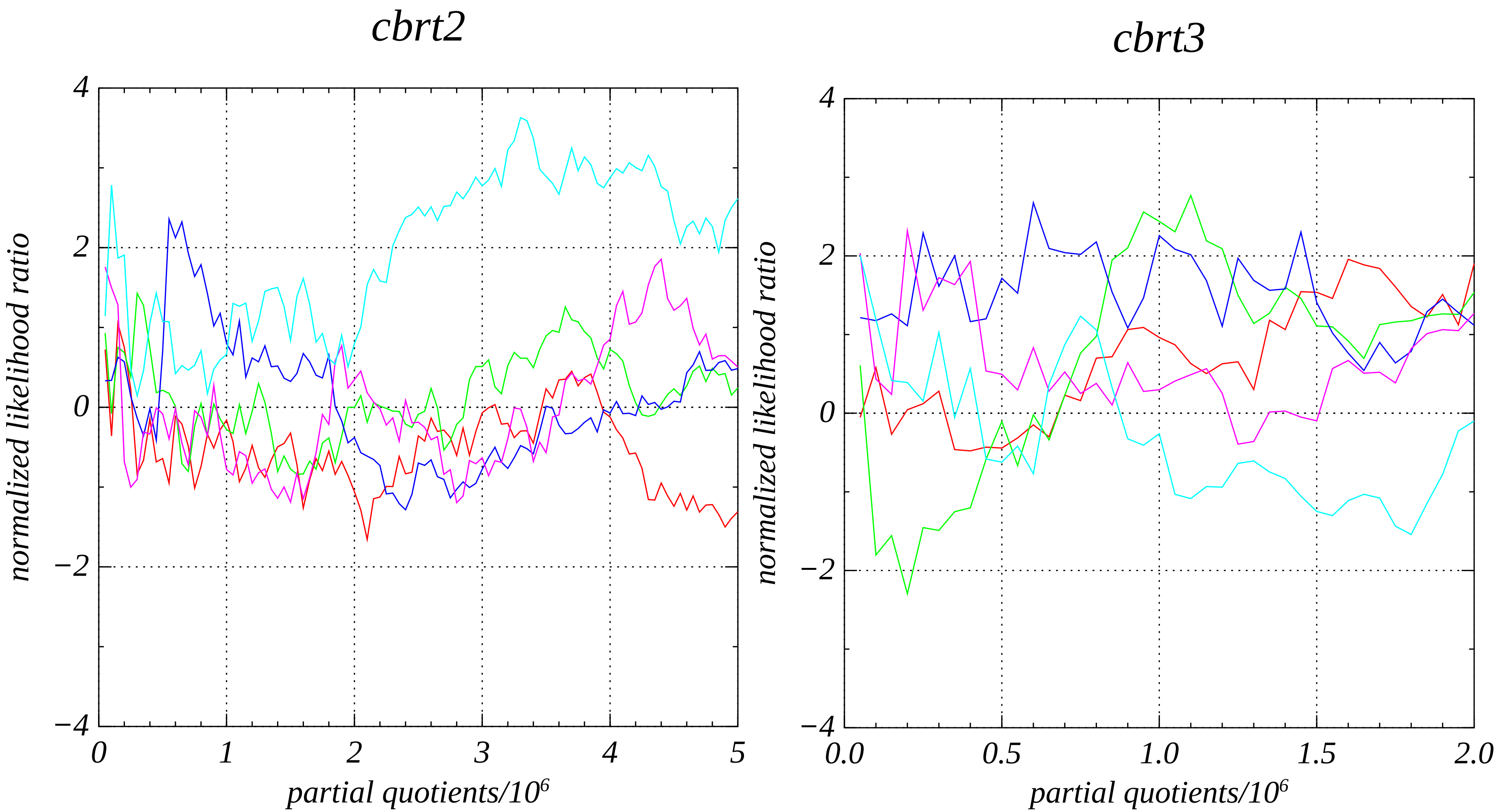

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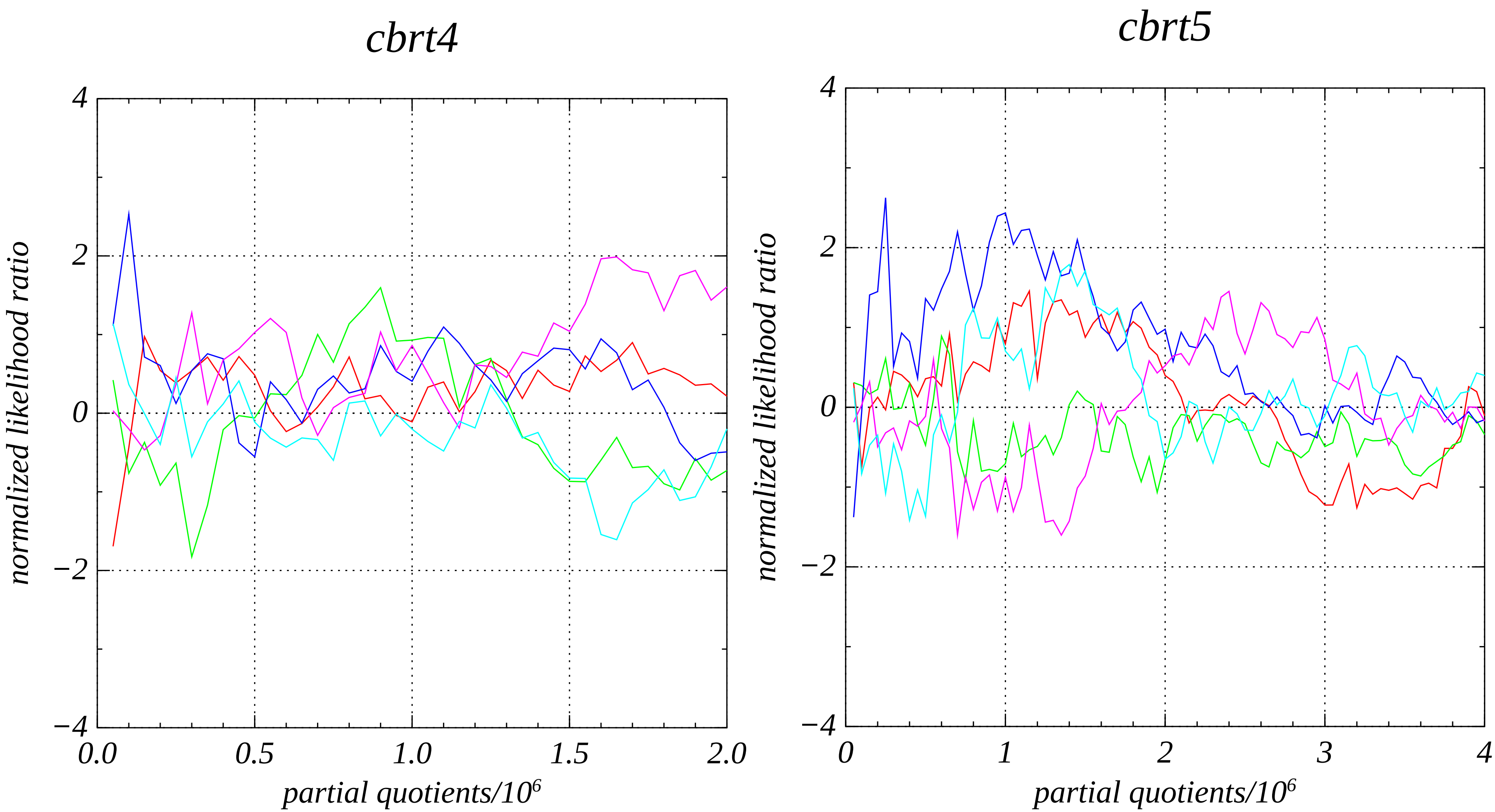

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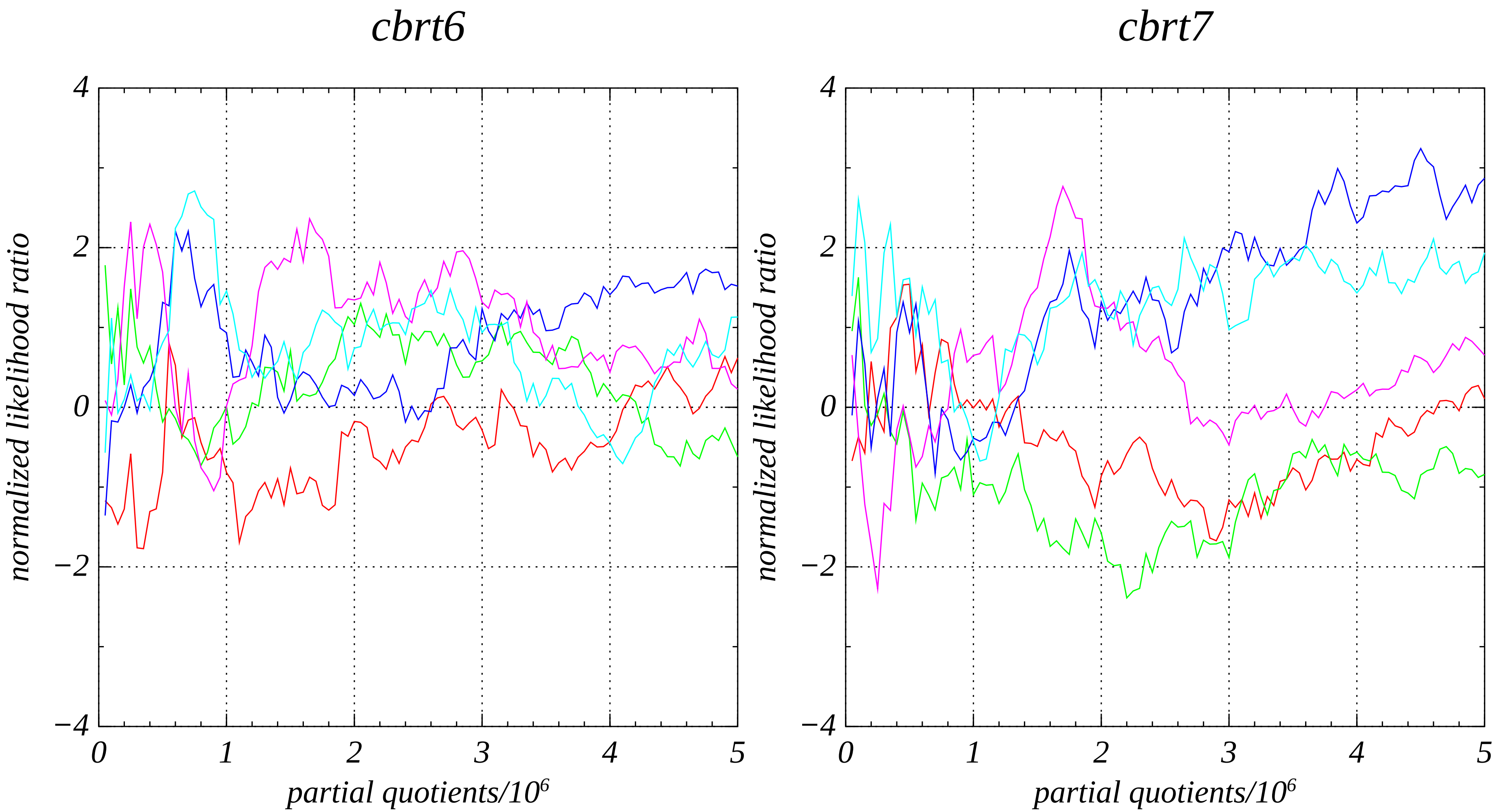

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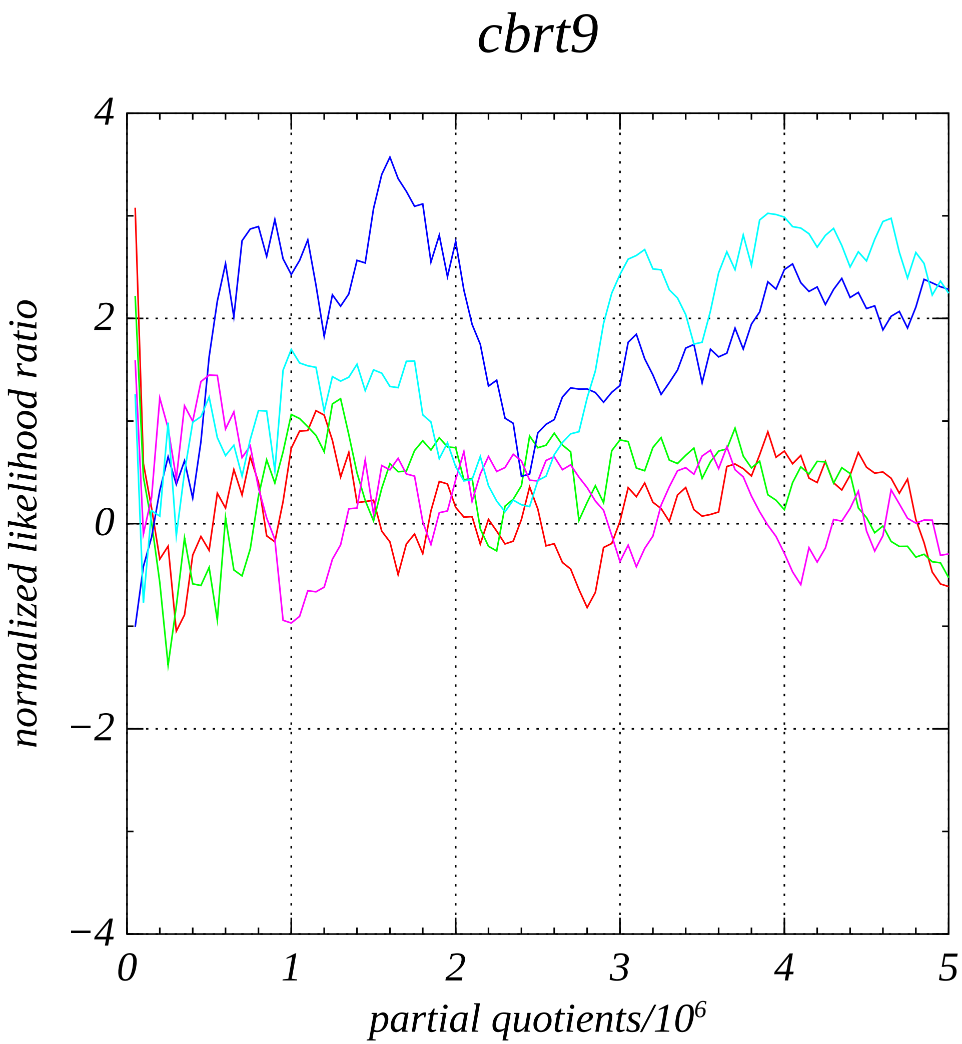

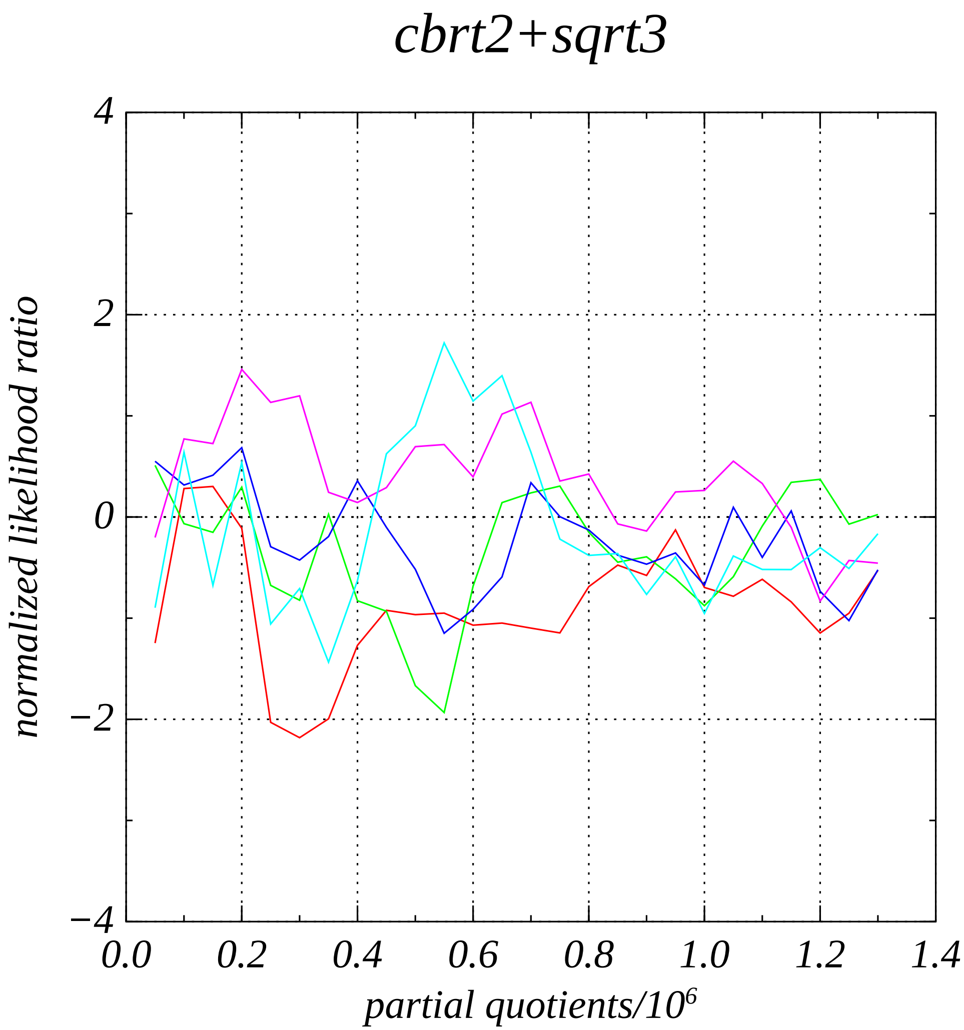

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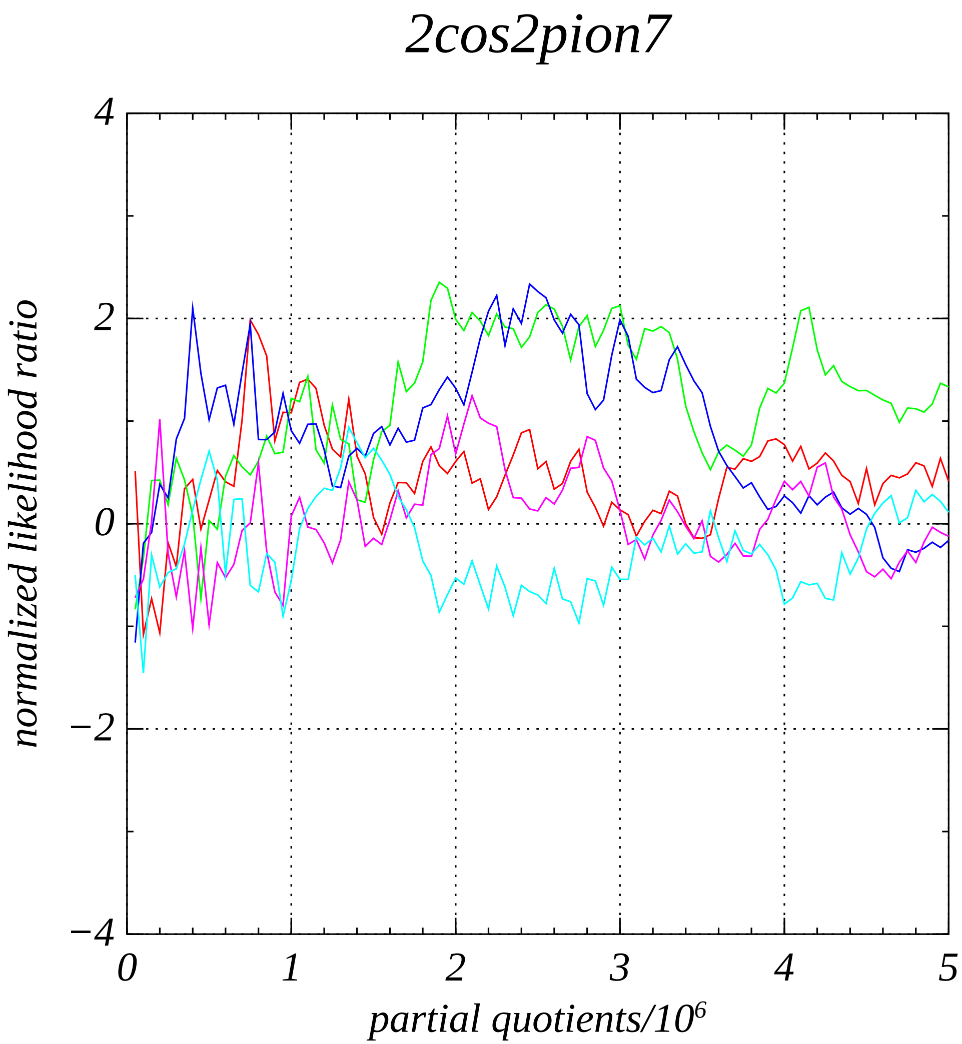

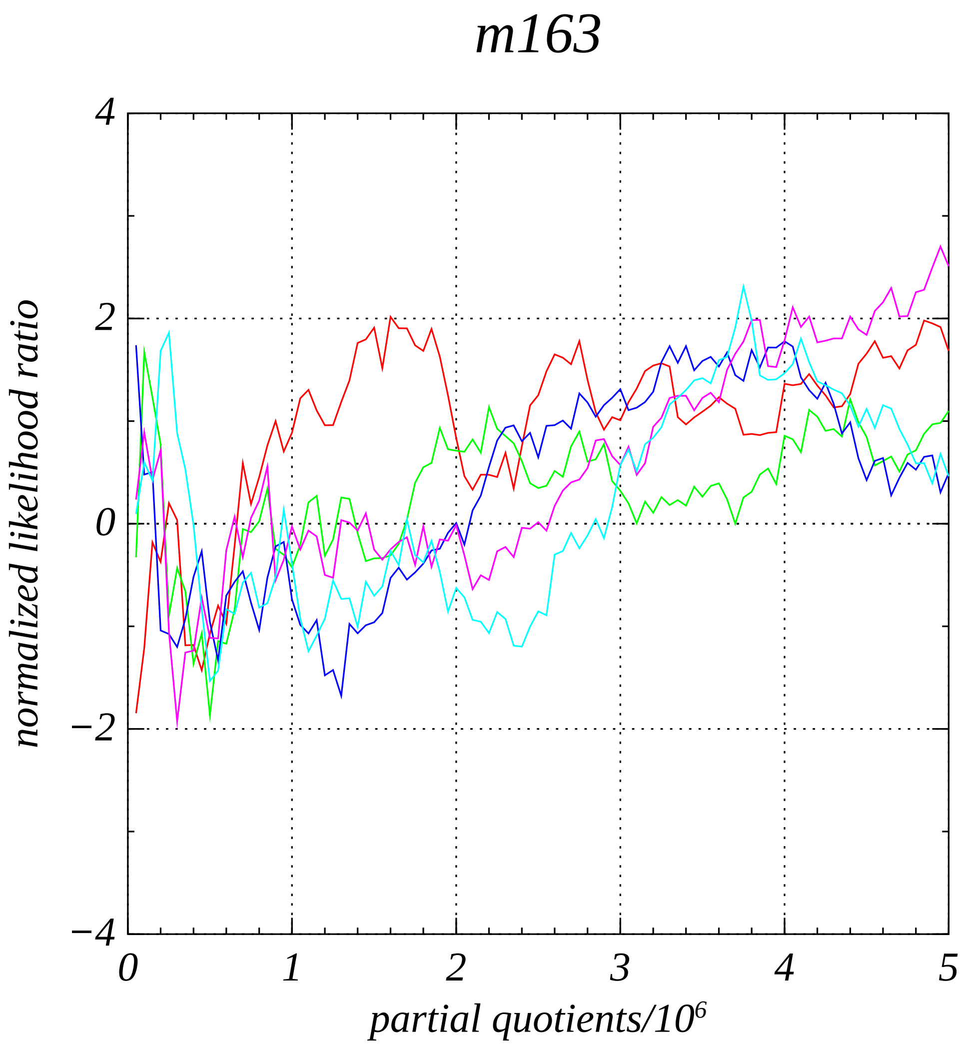

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Autocorrelation of digits

- We would expect the the autocorrelation function (acf) of any analytic function of the digits that has a finite mean (for example, the log or the reciprocal) would decay like q^k at lag k, where $q \approx -0.3$ is Wirsing's constant
- This is investigated in the following graphs. I plot log_{10} of the absolute value of the acf as a function of lag. The green line has the Wirsing slope

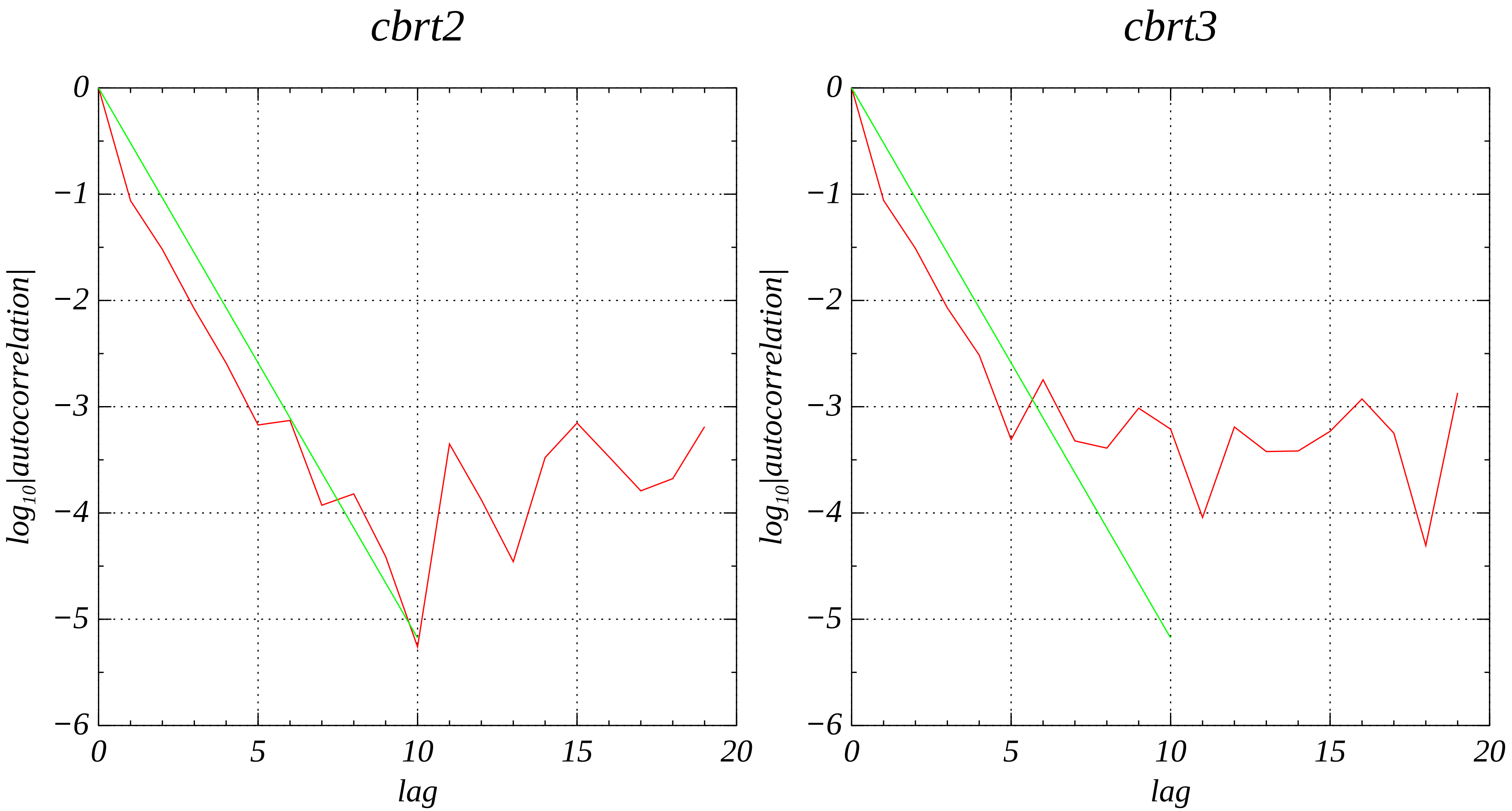
acf estimation difficulties

- ⋆ For the AR(1) process $x(t+1) = \alpha x(t) + \epsilon$, $|\alpha| < 1$, the exact acf at lag k is $\rho(k) = \alpha^k$
- ⋆ But the usual acf estimator r for a sample of size n has variance

$$\text{var}\left[r_n(k)\right] = \frac{1}{n}\left[\frac{(1+\alpha^2)(1+\alpha^{2k})}{1-\alpha^2} - 2k\alpha^{2k}\right]$$

- ⋆ More generally, for a process whose acf decays for large k in the same power-law fashion, we have approximate variance $\text{var}\left[r_n(k)\right] = \frac{1}{n}\left[\frac{1+\alpha^2}{1-\alpha^2}\right]$ for large k.
- ⋆ I expect my process to conform to this behaviour, and if it does, putting in the numbers gives an estimator of $k = 6$ for the largest k for which the acf estimates are meaningful

autocorrelation of logs of digits: $2^{1/3}$ and $3^{1/3}$



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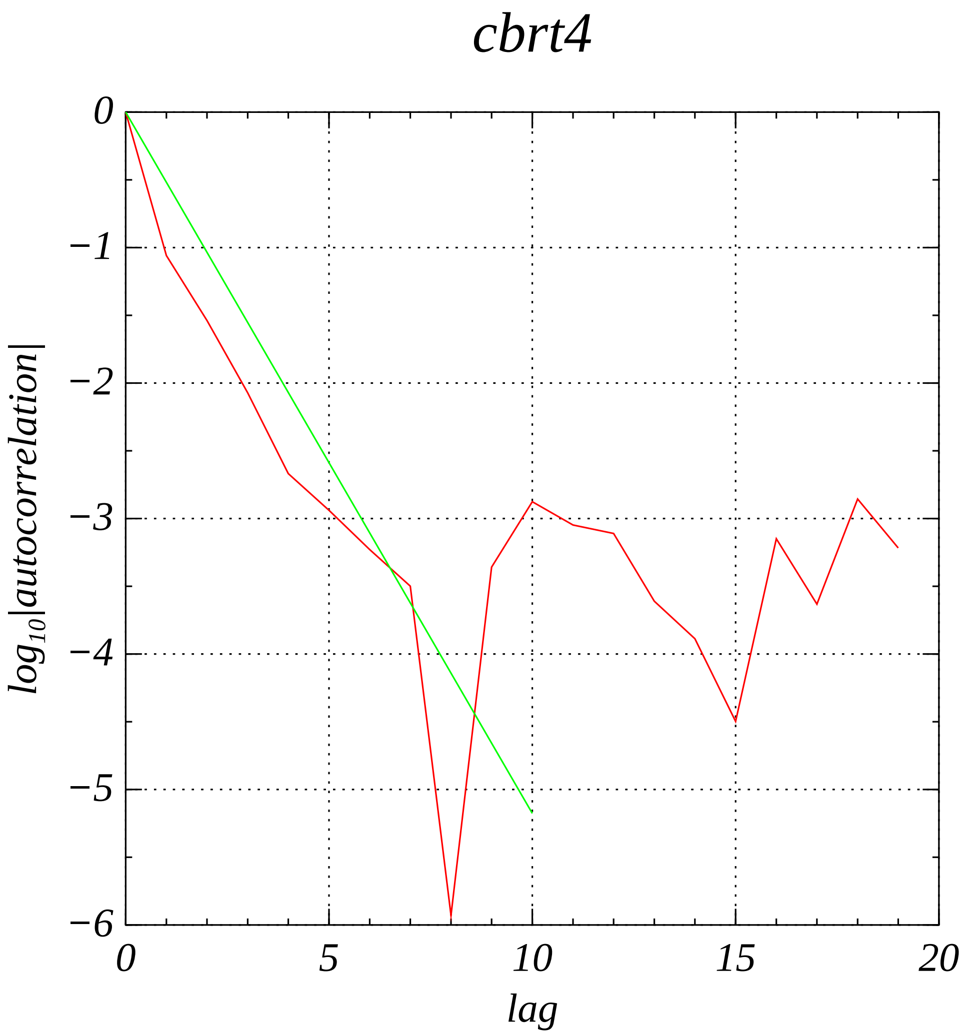

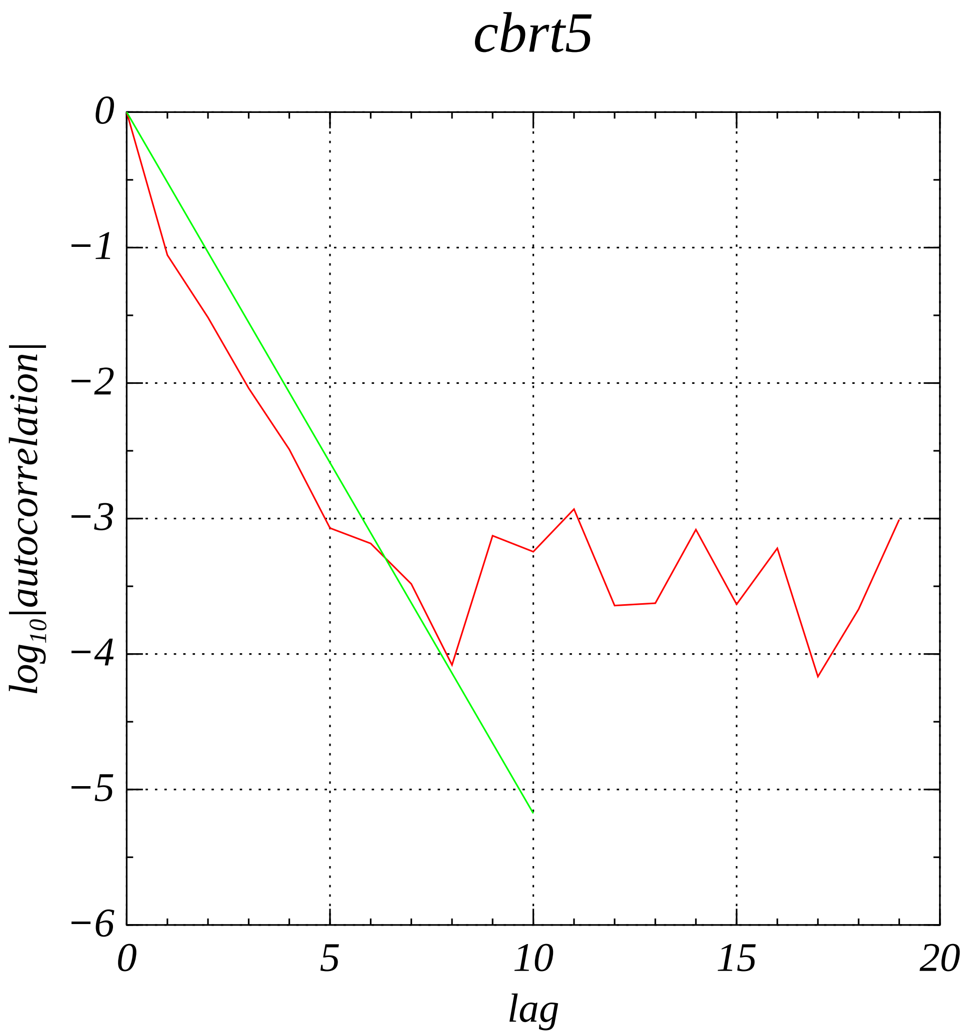

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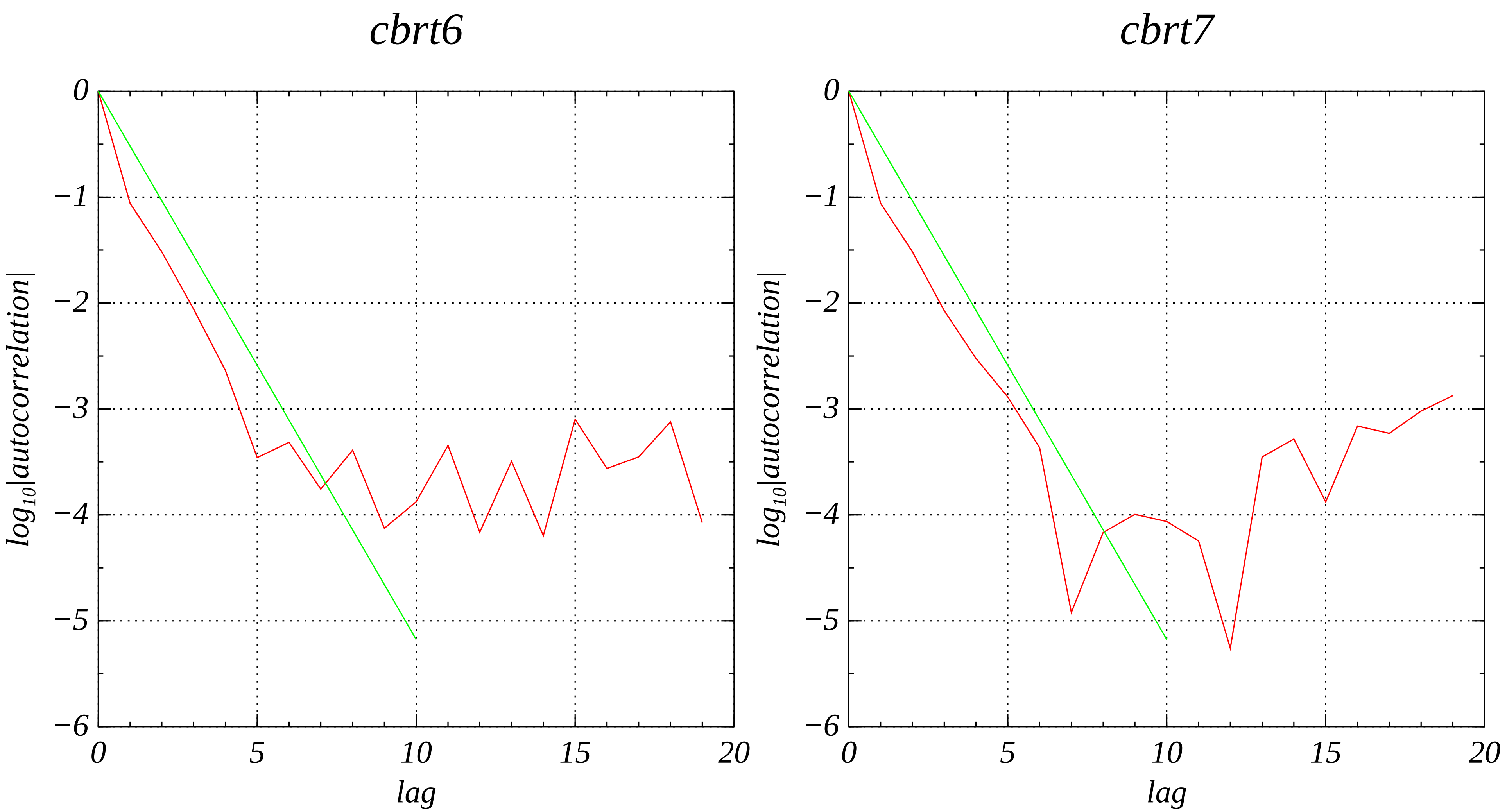

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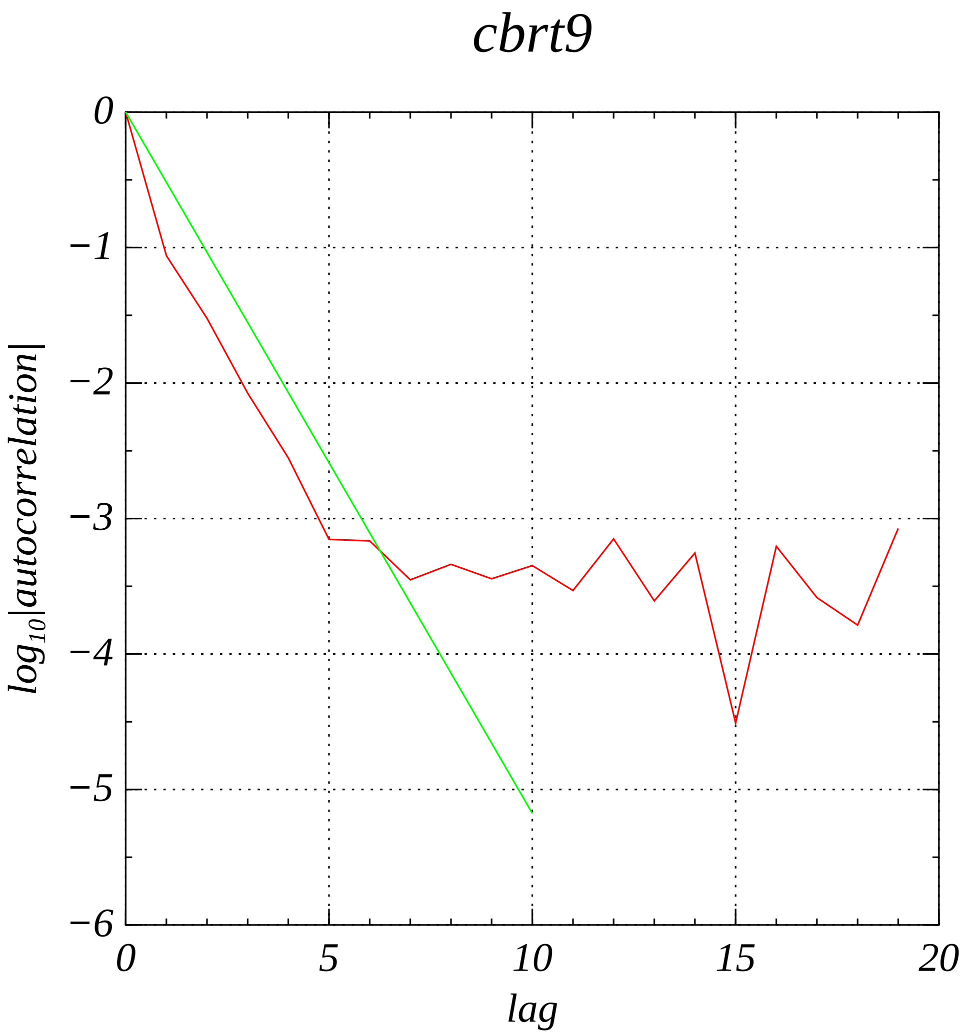

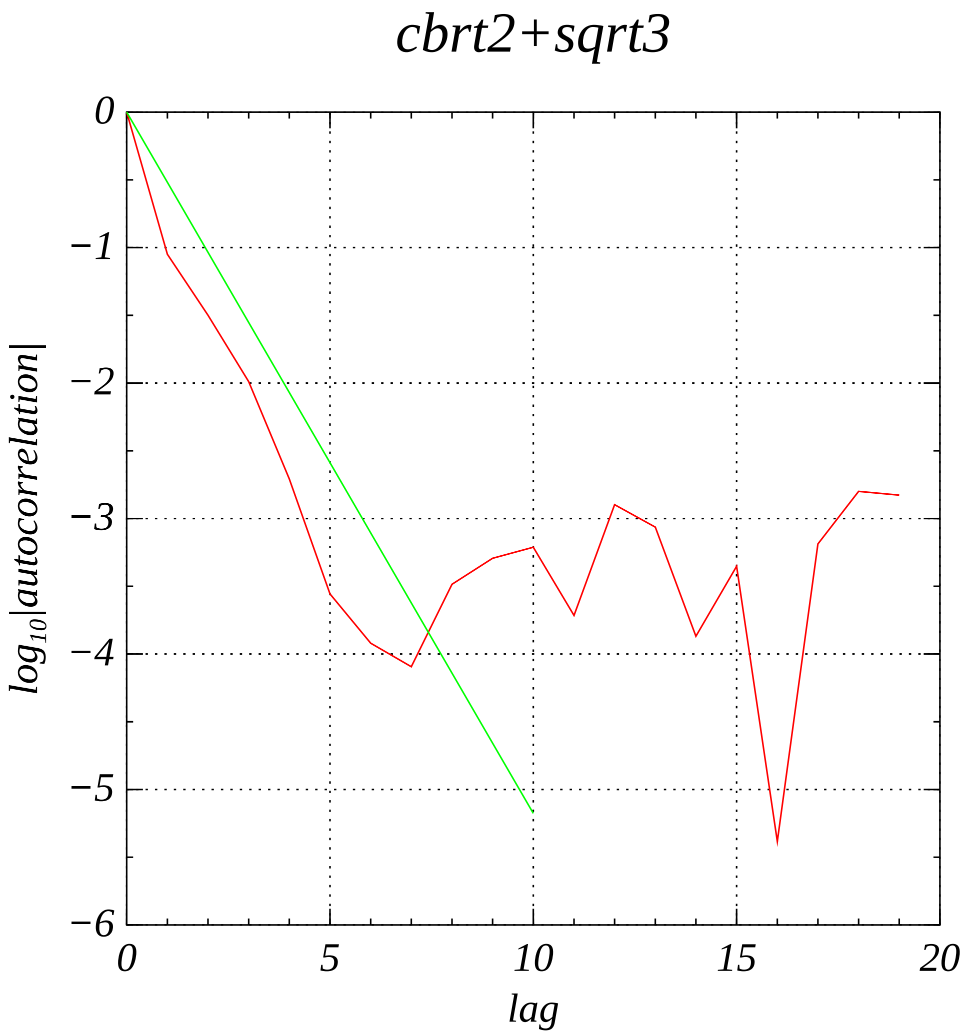

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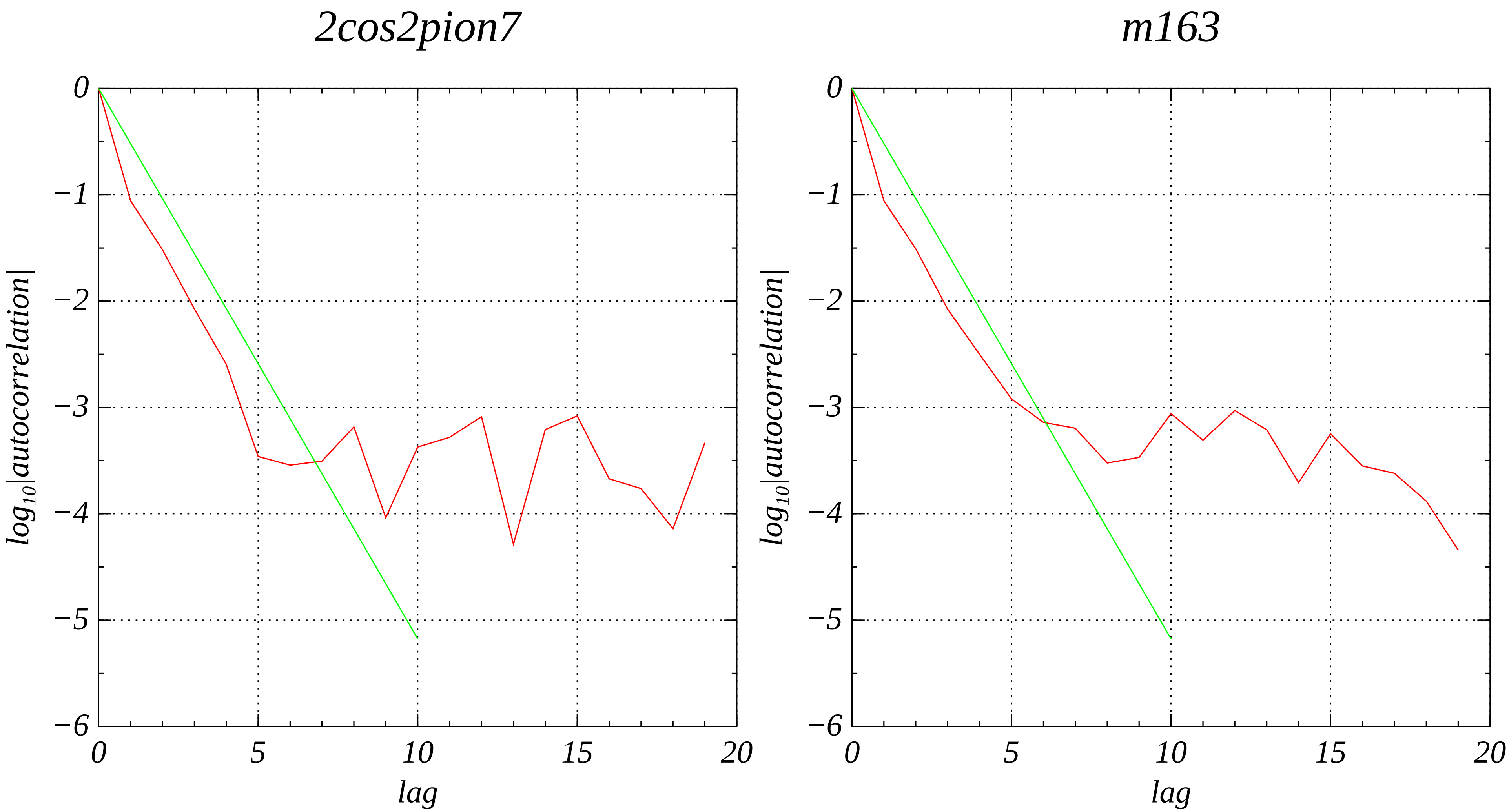

References

[1] R F Churchhouse & S T E Muir *Continued fractions, algebraic numbers and modular invariants*, J. Inst. Maths Applics **5**, 318-328 (1965)

[2] S Lang & H Trotter *Continued fractions for some algebraic numbers*, J. reine ang. Math. **255**, 112-134 (1972)

[3] R P Brent, A J van der Poorten & H J J te Riele *A comparative study of algorithms for computing continued fractions of algebraic numbers* in: Algorithmic Number Theory (H Cohen, ed.), LNCS, vol 1122, Springer-Verlag, 1996, 35-47

[4] D H Bailey, J M Borwein & R E Crandall *On the Khintchine constant*, Math. Comp. **66**, 417-431 (1997)

[5] D Hensley *Metric Diophantine approximation and probability*, New York J. Math. **4**, 249-257 (1998)

[6] D Hensley *The statistics of the continued fraction digit sum*, Pac. J. Math. **192**, 103-120 (2000)

[7] A M Rockett & P Szüsz *Continued Fractions*, World Scientific 1992, ISBN 981-02-1052-3

[8] M Iosifescu & C Kraaikamp *Metrical Theory of Continued Fractions*, Kluwer 2002, ISBN 1402008929 (Mathematics and Its Applications, vol 547)