

# Notes on combinatorial graph theory

**Keith Briggs**

Keith.Briggs@bt.com

[more.btexact.com/people/briggsk2/cgt.html](http://more.btexact.com/people/briggsk2/cgt.html)



CRG meeting 2004 Jan 19 Monday 15:00

comb-graph-th.tex TYPESET 2004 JANUARY 27 11:16 IN PDF $\text{\LaTeX}$  ON A LINUX SYSTEM

# The inspiration

➡ Bruno Salvy: Phénomène d'Airy et combinatoire analytique des graphes connexes (INRIA seminar 2003 Dec 15)

➡ [algo.inria.fr/seminars/seminars.html](http://algo.inria.fr/seminars/seminars.html)

➡ e.g. number of **labelled (étiquetés)** connected graphs: with excess (edges-vertices) =  $k \geq 1$  is

$$A_k(1) \sqrt{\pi} \binom{n}{e}^n \binom{n}{2}^{\frac{3k-1}{2}} \left[ \frac{1}{\Gamma(3k/2)} + \frac{A'_k(1)/A_k(1) - 1}{\Gamma((3k-1)/2)} \sqrt{2/n} + \mathcal{O}\left(\frac{1}{n}\right) \right]$$

➡  $A_k(1)$  given in terms of Airy functions:

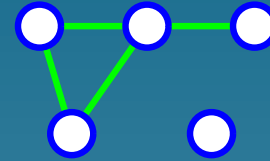
$$A_1(1) = 5/24, A_2(1) = 5/16 \text{ etc.}$$

➡ Airy in Playford:

[www.ast.cam.ac.uk/~ipswich/History/Airys\\_Country\\_Retreat.htm](http://www.ast.cam.ac.uk/~ipswich/History/Airys_Country_Retreat.htm)

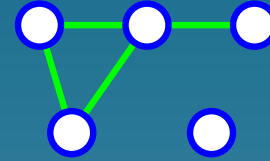
# Definitions for graphs

👉 (simple unlabelled undirected) graph:

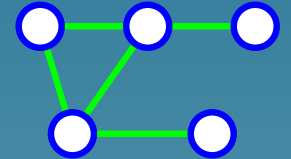


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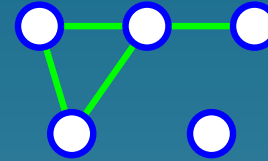


👉 (simple unlabelled undirected) **connected** graph:

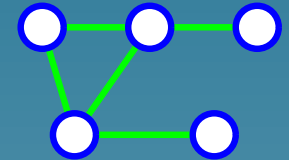


# Definitions for graphs

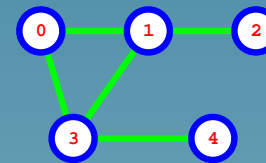
☞ (simple unlabelled undirected) graph:



☞ (simple unlabelled undirected) **connected** graph:



☞ (simple undirected) **labelled** graph:



# Definitions for generating functions

👉 generating function (gf):

$$\{a_1, a_2, a_3, \dots\} \leftrightarrow \sum_{k=1}^{\infty} a_k x^k$$

👉 exponential generating function (egf):

$$\{a_1, a_2, a_3, \dots\} \leftrightarrow \sum_{k=1}^{\infty} \frac{a_k}{k!} x^k$$

👉 Euler transform ( $b = ET(a)$ ):

$$1 + \sum_{k=1}^{\infty} b_k x^k = \prod_{i=1}^{\infty} (1 - x^i)^{-a_i} \leftrightarrow \log(1 + B(x)) = \sum_{k=1}^{\infty} A(x^k)/k$$

# Exponential generating functions

☞ exponential generating function for all labelled graphs:

$$g(w, z) = \sum_{n=0}^{\infty} (1+w)^{\binom{n}{2}} z^n / n!$$

☞ exponential generating function for all connected labelled graphs:

$$\begin{aligned} c(w, z) &= \log(g(w, z)) \\ &= z + w \frac{z^2}{2} + (3w^2 + w^3) \frac{z^3}{6} + (16w^3 + 15w^4 + 6w^5 + w^6) \frac{z^4}{4!} + \dots \end{aligned}$$

# egfs for labelled graphs [jan]

☞ rooted labelled trees

$$T(z) = z \exp(T(z)) = \sum_{n \geq 1} n^{n-1} \frac{z^n}{n!} = z + \frac{2}{2!}z^2 + \frac{9}{3!}z^3 + \dots$$

☞ unrooted labelled trees

$$U(z) = T(z) - T(z)^2/2 = z + \frac{1}{2!}z^2 + \frac{3}{3!}z^3 + \frac{16}{4!}z^4 + \dots$$

☞ unicyclic labelled graphs

$$\widehat{V}(z) = \frac{1}{2} \log \left[ \frac{1}{1-T(z)} \right] - \frac{1}{2}T(z) - \frac{1}{4}T(z)^2 = \frac{1}{3!}z^3 + \frac{15}{4!}z^4 + \frac{222}{5!}z^5 + \frac{3660}{6!}z^6 + \dots$$

☞ bicyclic labelled graphs

$$\widehat{W}(z) = \frac{T(z)^4(6-T(z))}{24(1-T(z))^3} = \frac{6}{4!}z^4 + \frac{205}{5!}z^5 + \frac{5700}{6!}z^6 + \dots$$



## Unlabelled graphs with $n$ nodes [slo]

👉 no simple exact formula available - use group theory

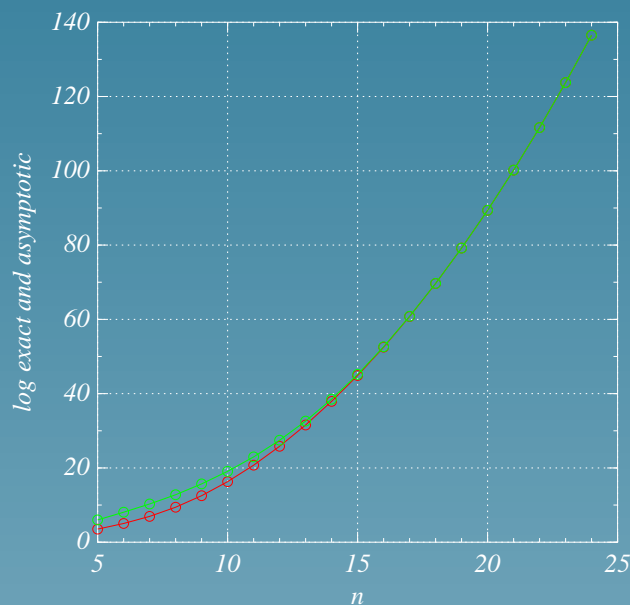
👉  $g = 1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, 165091172592, 50502031367952, 29054155657235488, 314264859698043$

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👉 
$$g_n = \frac{2^{\binom{n}{2}}}{n!} \left[ 1 + \frac{n(n-1)}{2^{n-1}} + \frac{8n!}{2^{2n}(n-4)!} (3n-7)(3n-9) + \mathcal{O}(n^5/2^{5n/2}) \right]$$



exact asymptotic

## Unlabelled connected graphs with $n$ nodes [slo03]

☞  $c = 1, 1, 1, 2, 6, 21, 112, 853, 11117, 261080, 11716571, 1006700565,$   
 $164059830476, 50335907869219, 29003487462848061,$   
 $31397381142761241960, 63969560113225176176277,$   
 $245871831682084026519528568, 1787331725248899088890200576580,$   
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☞  $g = ET(c)$

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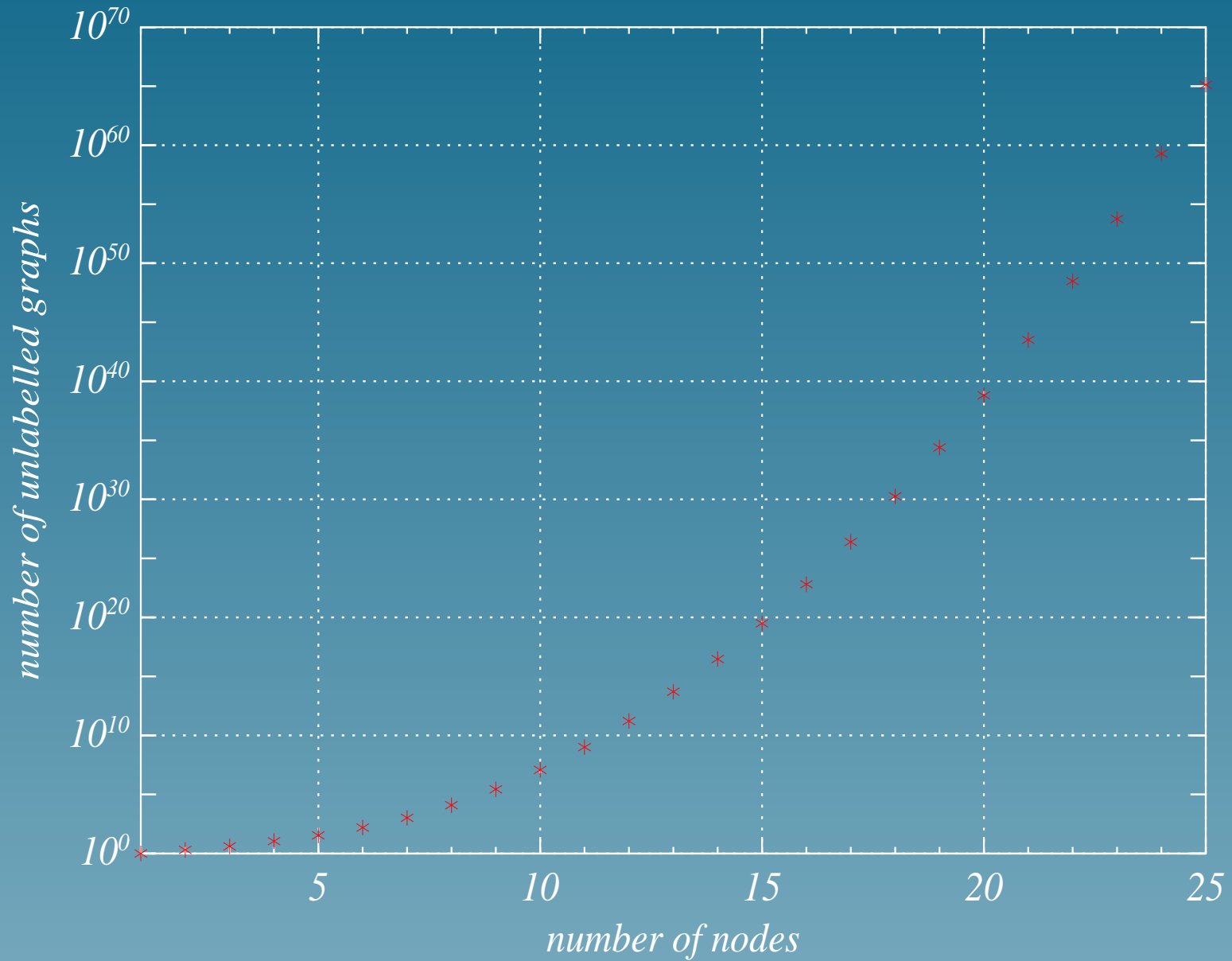
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☞  $g = ET(c)$

☞ Edge generating functions can be computed: e.g. for  $n = 5$ :

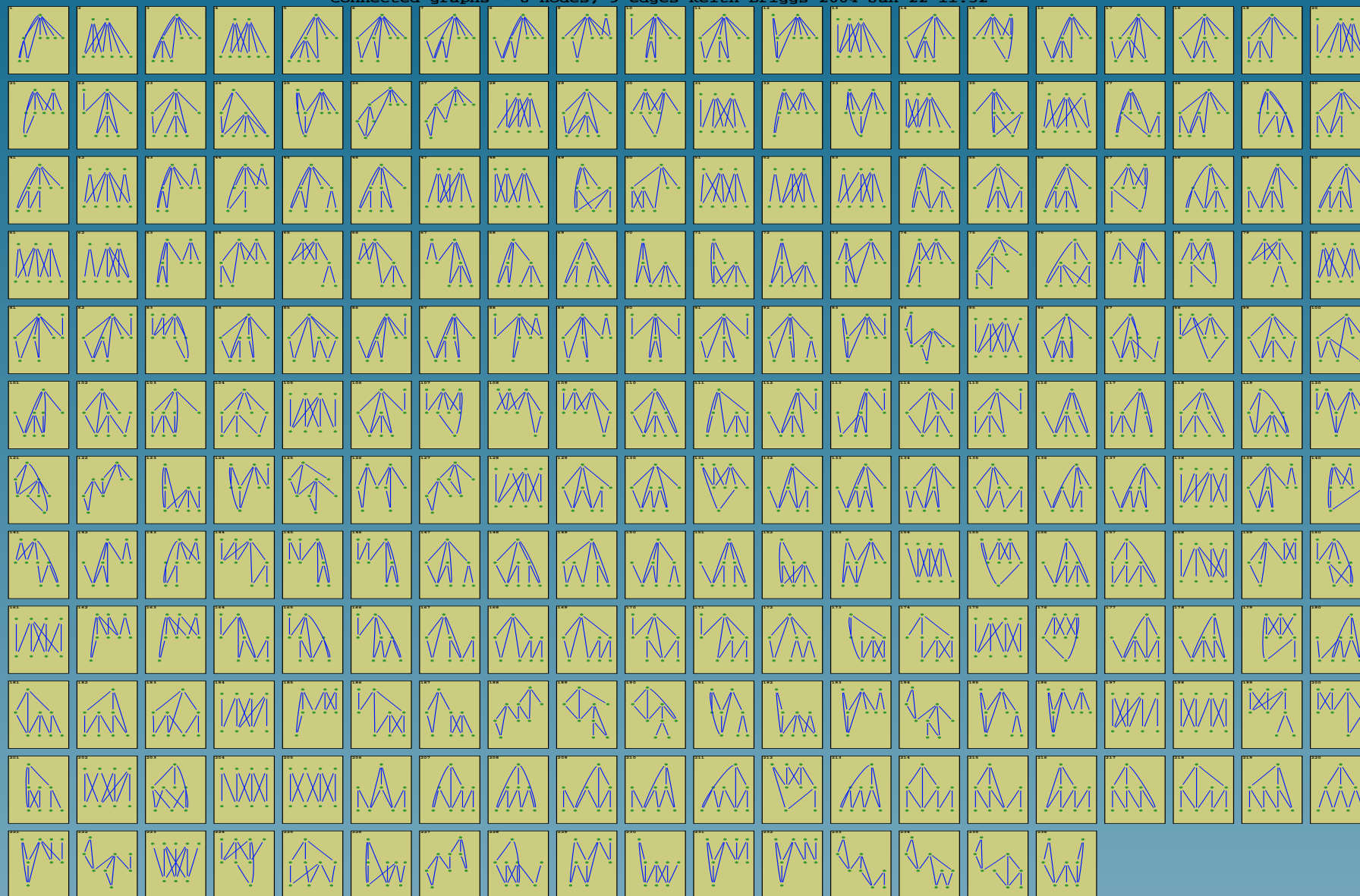
$$1 + q + q^2 + 2q^3 + 6q^4 + 6q^5 + 6q^6 + 4q^7 + 2q^8 + q^9 + q^{10}$$

# Total numbers of unlabelled graphs



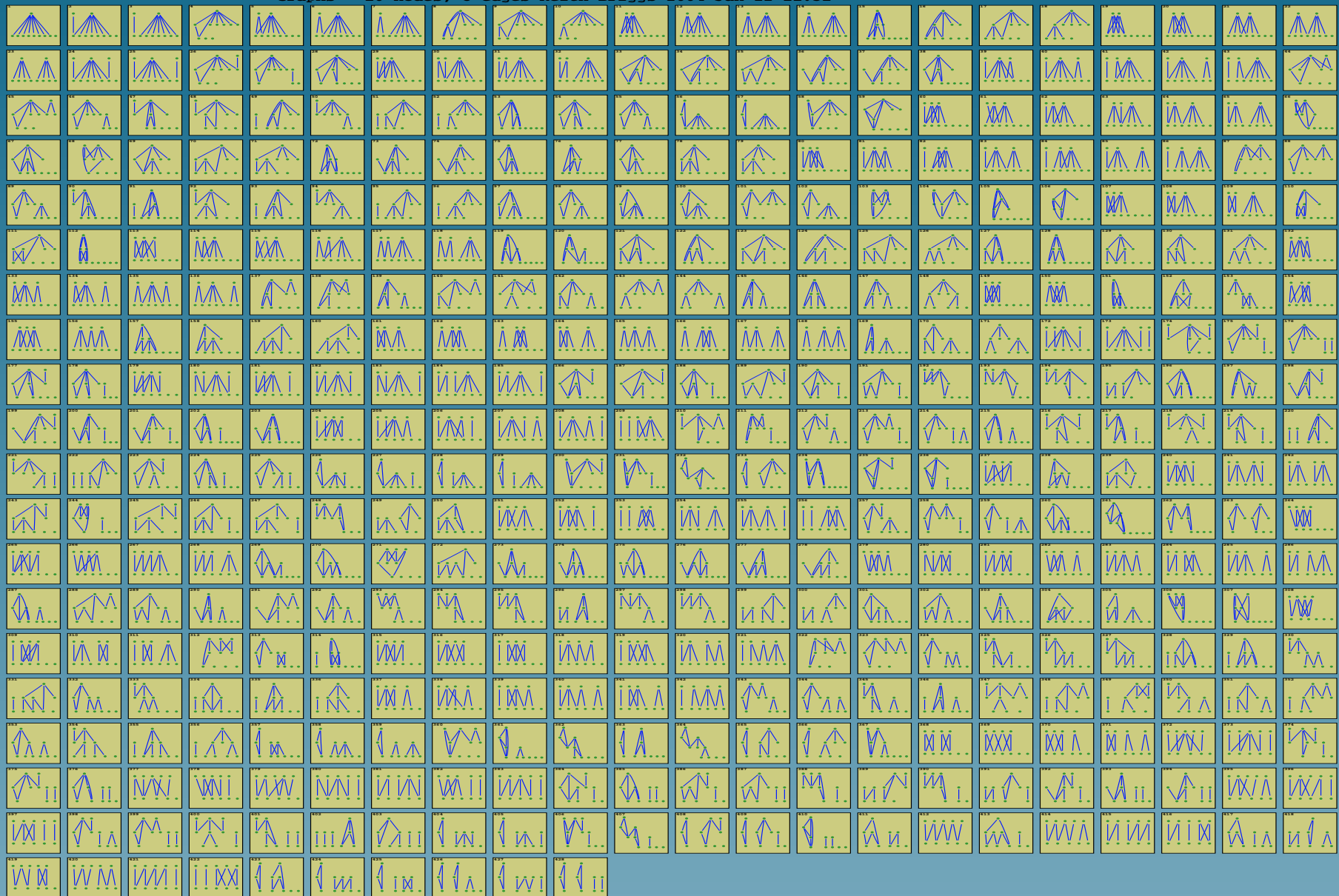
# Connected unlabelled graphs - 8 nodes and 9 edges

Connected graphs - 8 nodes, 9 edges Keith Briggs 2004 Jan 22 11:32



# Unlabelled graphs - 10 nodes and 8 edges

Graphs - 10 nodes, 8 edges Keith Briggs 2004 Jan 22 11:31





## Probability of connectivity 1 [gil]

☞ Bernoulli random graph model of Erdős and Rényi: edges appear independently with probability  $p = 1 - q$ . Let  $P(n, p) = 1 - Q(n, p)$  be the probability that such a graph with  $n$  labelled nodes is connected.

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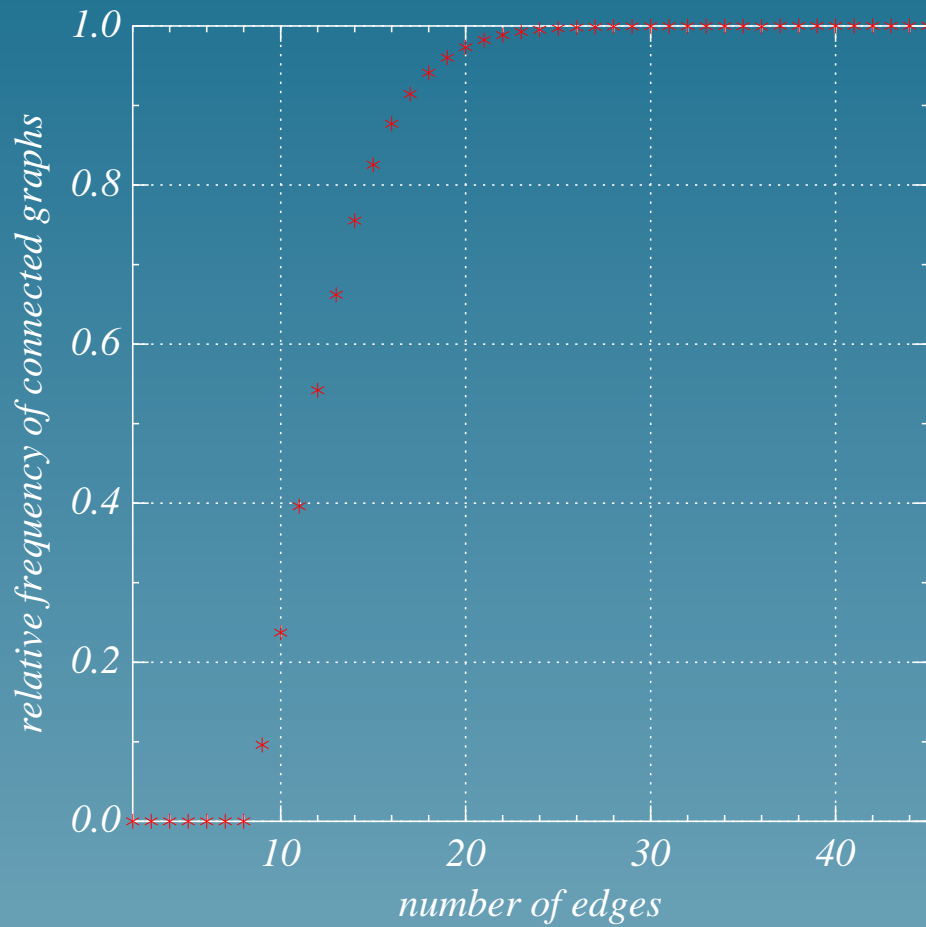
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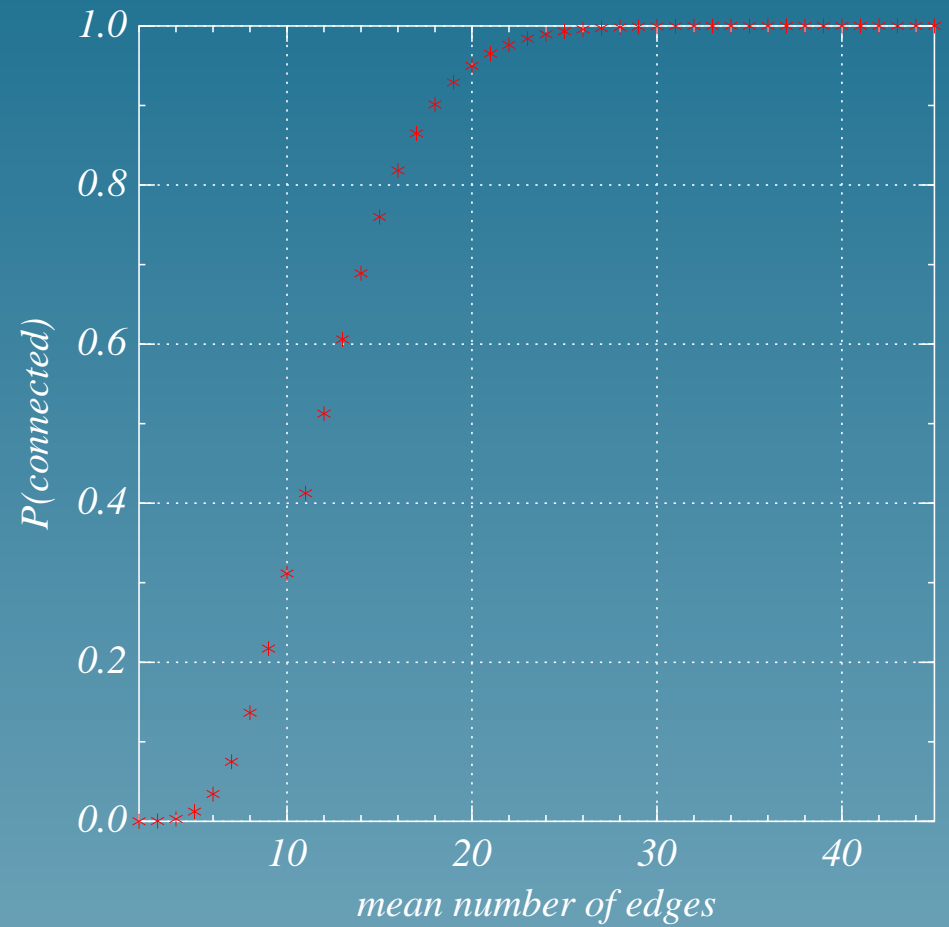
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- ➡  $P(n, p) \sim 1 - n q^{n-1}$  as  $n \rightarrow \infty$

# Probability of connectivity 2

*Exact enumeration, 10 unlabelled nodes*



*Bernoulli rg model, 10 labelled nodes*



## References

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