Enumerating connected labelled graphs

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The problem

- compute the numbers of connected labelled graphs with n nodes and $n-1, n, n+1, n+2, \ldots$ edges
  - *with this information, we can compute the probability of a randomly chosen labelled graph being connected*
- compute large-n asymptotics for these quantities, where the number of edges is only slightly larger than the number of nodes
- I began by reading the paper [fss04], but found some inconsistencies
- so I did some exact numerical calculations to try to establish the dominant asymptotics
- I then looked at some earlier papers and found that the required theory to compute exact asymptotics *is* known
- I computed the exact asymptotics and got perfect agreement with my exact numerical data

The paper [fss04]

- Philippe Flajolet, Bruno Salvy and Gilles Schaeffer: *Airy Phenomena and Analytic Combinatorics of Connected Graphs* `www.combinatorics.org/Volume_11/Abstracts/v11i1r34.html`

- The claim: the number $C(n, n+k)$ of labelled (étiquetés) connected graphs with n nodes and excess (edges-nodes) $= k \geqslant 2$ (why not for $k=1$?) is

$$A_k(1)\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(\frac{n}{2}\right)^{\frac{3k-1}{2}}\left[\frac{1}{\Gamma(3k/2)}+\frac{A_k'(1)/A_k(1)-k}{\Gamma((3k-1)/2)}\sqrt{\frac{2}{n}}+\mathcal{O}\left(\frac{1}{n}\right)\right]$$

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k	1	2	3	4	5	6	7	8
$A_k(1)$	5/24	5/16	1105/1152	565/128	82825/3072	19675/96	1282031525/688128	8
$A_k'(1)$	19/24	65/48	1945/384	21295/768	603965/3072	10454075/6144	1705122725/98304	3

- Airy in Playford: `www.ast.cam.ac.uk/~ipswich/History/Airys_Country_Retreat.htm`

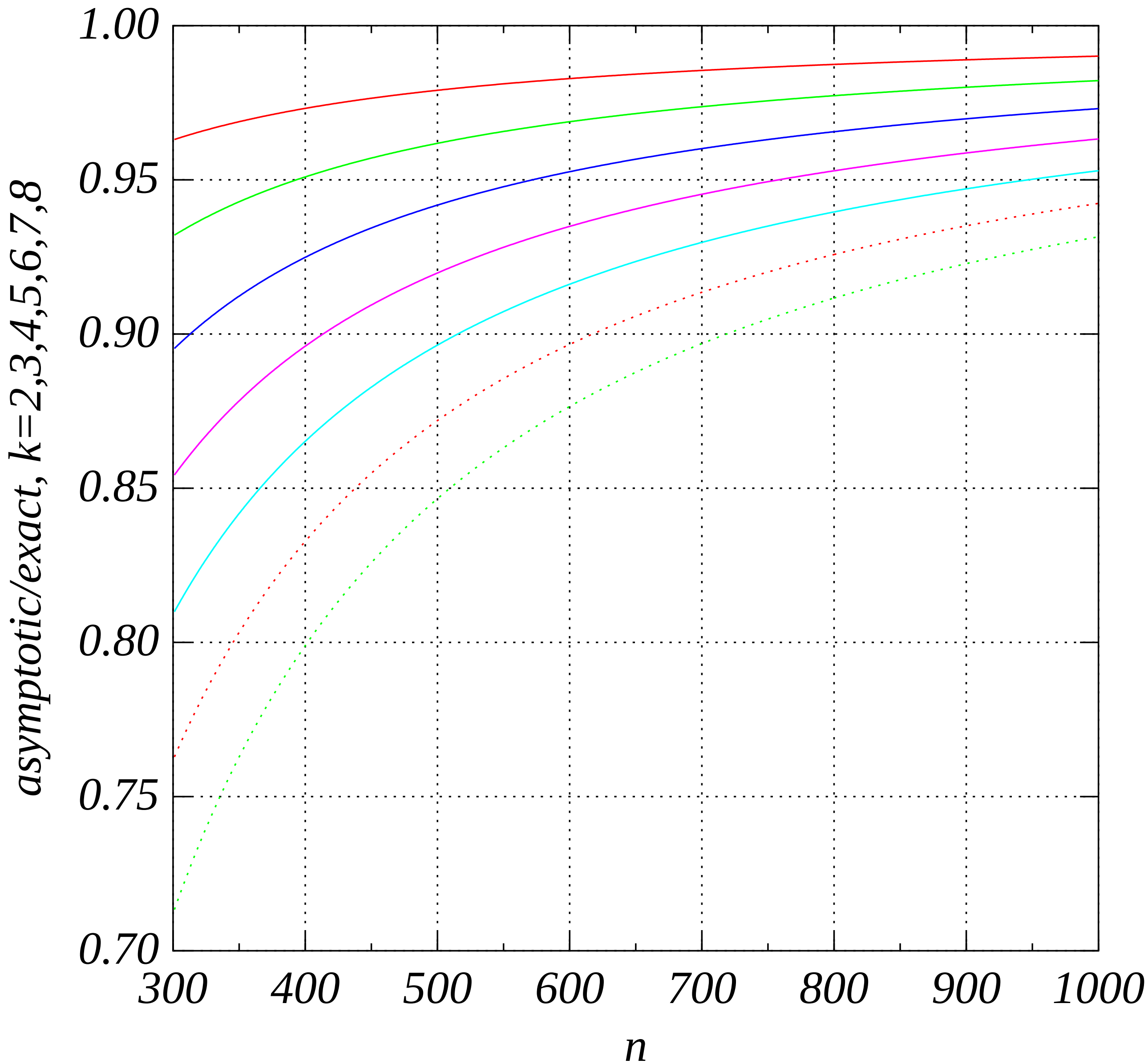
Some small problems?

- ☞ I did some comparisons with exact counts for up to $n = 1000$ nodes and for excess $k = 2, 3, \ldots, 8$
- ☞ The exact data was computed from the generating functions using maxima (found to be faster than maple)
- ☞ The fit was very bad
- ☞ This formula was found to fit the data much better for $k = 2$:

$$A_k(1)\sqrt{\pi}n^n\left(\frac{n}{2}\right)^{\frac{3k-1}{2}}\left[\frac{1}{\Gamma(3k/2)} - \frac{A_k'(1)/A_k(1)-k}{\Gamma((3k-1)/2)}\sqrt{\frac{2}{n}}+\mathcal{O}\left(\frac{1}{n}\right)\right]$$

- ☞ Also, on pages 4 and 24, I think S should have the expansion $1-(5/4)\alpha+(15/4)\alpha^2+\ldots$

Comparison of exact data with corrected formula



Asymptotic expansion of $C(n,n+k)/n^{n+\frac{3k-1}{2}}$

$\xi \equiv \sqrt{2\pi}$ green: from [bcm90] red: from [fss04] (with removal of factor e)

k	type	$[n^0]$	$[n^{-1/2}]$	$[n^{-1}]$	$[n^{-3/2}]$
-1	tree	1	0	0	0
0	unicycle	$\xi\frac{1}{4}$			
1	bicycle	$\frac{5}{24}$			
2	tricycle	$\xi\frac{5}{256}$ $\xi\frac{5}{256}$	$-\frac{35}{144}$		
3	quadricycle	$\frac{221}{1512}$ $\frac{221}{24192}$	$-\sqrt{\pi}\frac{35}{96}$		
4	pentacycle	$\xi\frac{113}{196608}$			

blue: conjectured by KMB from numerical experiments

k	type	$[n^0]$	$[n^{-1/2}]$	$[n^{-1}]$	$[n^{-3/2}]$	$[n^{-2}]$	$[n^{-5/2}]$
0	unicycle	$\xi\frac{1}{4}$	$-\frac{7}{6}$	$\xi\frac{1}{48}$	$\frac{131}{270}$	$\xi\frac{1}{1152}$	$-\frac{4}{2835}$?
1	bicycle	$\frac{5}{24}$	$-\xi\frac{7}{24}$	$\frac{25}{36}$	$-\xi\frac{7}{288}$	$-\frac{79}{3240}$?	
2	tricycle	$\xi\frac{5}{256}$	$-\frac{35}{144}$	$\xi\frac{1559}{9216}$	$-\frac{55}{144}$		
3	quadricycle	$\frac{221}{24192}$	$-\xi\frac{35}{10706}$				

Theory 1

The previous observations can be proved using theory available in [jklp93] and [fgkp95]. I sketch the computations.

- ☞ Ramanujan's Q-function is defined for $n = 1, 2, 3, \ldots$: $Q(n) \equiv \sum_{k=1}^{\infty} \frac{n^{\underline{k}}}{n^k} = 1 + \frac{n-1}{n} + \frac{(n-1)(n-2)}{n^2} + \ldots,$
- ☞ $\sum_{n=1}^{\infty} Q(n) n^{n-1} \frac{z^n}{n!} = -\log(1 - T(z))$, where T is the egf for rooted labelled trees: $T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n$
- ☞ $T(z) = z \exp(T(z))$

Theory 2

☞ to get the large-n asymptotics of Q, we first consider the related function $R(n)\equiv 1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\ldots,\ n=1,2,3,\ldots$

- ▷ *we have* $Q(n)+R(n)=n!\,e^n/n^n$
- ▷ *let* $D(n)=R(n)-Q(n)$
- ▷ $\sum_{n=1}^{\infty} D(n)n^{n-1}\frac{z^n}{n!}=\log[\frac{(1-T(z))^2}{2(1-ez)}]$
- ▷ $D(n)\sim\sum_{k=1}^{\infty} c(k)[z^n](T(z)-1)^k$, *where* $c(k)\equiv[\delta^k]\log(\delta^2/2/(1-(1+\delta)e^{-\delta}))$
- ▷ *maple gives* $D(n)\sim\frac{2}{3}+\frac{8}{135}\,n^{-1}-\frac{16}{2835}\,n^{-2}-\frac{32}{8505}\,n^{-3}+\frac{17984}{12629925}\,n^{-4}+\frac{668288}{492567075}\,n^{-5}+O\left(n^{-6}\right)$

☞ now using $Q(n)=(n!\,e^n/n^n-D(n))/2$, we get

- ▷ $Q(n)\sim\frac{1}{2}n^{1/2}\sqrt{2\pi}-\frac{1}{3}+\frac{1}{24}\sqrt{2\pi}n^{-1/2}-\frac{4}{135}\,n^{-1}+\frac{1}{576}\sqrt{2\pi}n^{-3/2}+\frac{8}{2835}\,n^{-2}-\frac{139}{103680}\sqrt{2\pi}n^{-5/2}+\frac{16}{8505}\,n^{-3}-\frac{571}{4976640}\sqrt{2\pi}n^{-7/2}-\frac{8992}{12629925}\,n^{-4}+\frac{163879}{418037760}\sqrt{2\pi}n^{-9/2}-\frac{334144}{492567075}\,n^{-5}+\frac{5246819}{150493593600}\sqrt{2\pi}n^{-11/2}+O\left(n^{-6}\right)$

Theory 3

- ☞ Let W_k be the egf for connected labelled $k+1$-cyclic graphs
  - ▷ *for unrooted trees* $W_{-1}(z)=T(z)-T^2(z)/2$, $[z^n]W_{-1}(z)=n^{n-2}$
  - ▷ *for unicycles* $W_0(z)=-(\log(1-T(z))+T(z)+T^2(2)/2)/2$
  - ▷ *for bicycles* $W_1(z)=\frac{6T^4(z)-T^5(z)}{24(1-T(z))^3}$
  - ▷ *for* $k\geqslant 1$, $W_k(z)=\frac{A_k(T(z))}{(1-T(z))^{3k}}$, *where* A_k *are polynomials computable from results in [jklp93]*

- ☞ Knuth and Pittel's tree polynomials $t_n(y)$ ($y\neq 0$) are defined by $(1-T(z))^{-y}=\sum_{n=0}^{\infty}t_n(y)\frac{z^n}{n!}$
  - ▷ *we can compute these for* $y>0$ *from*
    $t_n(1)=1;\quad t_n(2)=n^n(1+Q(n));\quad t_n(y+2)=n\,t_n(y)/y+t_n(y+1)$

- ☞ thanks to this recurrence, the asymptotics for t_n follow from the known asymptotics of Q

Theory 4

We now have all the pieces needed. Let $\xi = \sqrt{2\pi}$. The asymptotic expansions below are computed by maple using results on the previous pages. All results agree with numerical estimates on this page.

- the number of connected unicycles is $C(n,n) = n![z^n]W_0(z) = \frac{1}{2}Q(n)n^{n-1} + 3/2 + t_n(-1) - t_n(-2)/4$

  ▷ $n^n \left(\frac{1}{4}\,\xi n^{-1/2} - \frac{7}{6}\,n^{-1} + \frac{1}{48}\,\xi n^{-3/2} + \frac{131}{270}\,n^{-2} + \frac{1}{1152}\,\xi n^{-5/2} + \frac{4}{2835}\,n^{-3} - \frac{139}{207360}\,\xi n^{-7/2} + \frac{8}{8505}\,n\right.$

- the number of connected bicycles is $C(n,n+1) = n![z^n]W_1(z) = \frac{5}{24}t_n(3) - \frac{19}{24}t_n(2) + \frac{13}{12}t_n(1) - \frac{7}{12}t_n(0) + \frac{1}{24}t_n(-1) + \frac{1}{24}t_n(-2)$

  ▷ $n^n \left(\frac{5}{24}\,n - \frac{7}{24}\,\xi n^{1/2} + \frac{25}{36} - \frac{7}{288}\,\xi n^{-1/2} - \frac{79}{3240}\,n^{-1} - \frac{7}{6912}\,\xi n^{-3/2} - \frac{413}{4860}\,n^{-2} + \frac{973}{1244160}\,\xi n^{-5/2} -\right.$

- similarly, for the number of connected tricycles we get

  ▷ $n^n \left(\frac{5}{256}\,\xi n^{5/2} - \frac{35}{144}\,n^2 + \frac{1559}{9216}\,\xi n^{3/2} - \frac{55}{144}\,n + \frac{33055}{221184}\,\xi n^{1/2} - \frac{41971}{136080} + \frac{31357}{2654208}\,\xi n^{-1/2} + \frac{1129}{81648}\,n\right.$

Probability of connectivity 1

☞ we now have all the results needed to calculate the asymptotic probability $P(n,n+k)$ that a randomly chosen graph with n nodes and $n+k$ edges is connected (for $n\to\infty$ and small fixed k)

☞ the total number of graphs is $g(n,n+k)\equiv\binom{\binom{n}{2}}{n+k}$. This can be asymptotically expanded:

- $\frac{g(n,n-1)}{\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{-3/2}} \sim 1+\frac{7}{4}n^{-1}+\frac{259}{96}n^{-2}+\frac{22393}{5760}n^{-3}+\frac{54359}{10240}n^{-4}+\frac{52279961}{7741440}n^{-5}+\frac{777755299}{103219200}n^{-6}+O\left(n^{-7}\right)$
- $\frac{g(n,n+0)}{\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{-1/2}} \sim \frac{1}{2}-\frac{5}{8}n^{-1}-\frac{53}{192}n^{-2}-\frac{4067}{11520}n^{-3}-\frac{9817}{20480}n^{-4}-\frac{10813867}{15482880}n^{-5}-\frac{217565701}{206438400}n^{-6}-\frac{11591924473}{7431782400}n^{-7}+O\left(n^{-8}\right)$
- $\frac{g(n,n+1)}{\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{3/2}} \sim \frac{1}{4}-\frac{21}{16}n^{-1}+\frac{811}{384}n^{-2}-\frac{43187}{23040}n^{-3}+\frac{159571}{73728}n^{-4}-\frac{55568731}{30965760}n^{-5}+\frac{2867716177}{1238630400}n^{-6}-\frac{3215346127}{2123366400}n^{-7}+\frac{1317595356557}{475634073600}n^{-8}+O\left(n^{-9}\right)$
- ...
- $g(n,n+k)\sim\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{k-1/2}\left(2^{-k-1}+O(n^{-1})\right)$

Probability of connectivity 2

☞ $P(n,n-1)=2^n e^{2-n} n^{-1/2}\xi\left(\frac{1}{2}-\frac{7}{8}n^{-1}+\frac{35}{192}n^{-2}+\frac{1127}{11520}n^{-3}+\frac{5189}{61440}n^{-4}+\right.$

- ▷ *check:* $n=10$, *exact*=0.1128460393, *asymptotic*=0.1128460359

☞ $P(n,n+0)=2^n e^{2-n}\xi\left(\frac{1}{4}\xi-\frac{7}{6}n^{-1/2}+\frac{1}{3}\xi n^{-1}-\frac{1051}{1080}n^{-3/2}+\frac{5}{9}\xi n^{-2}+O\left(n^{-}\right.\right.$

- ▷ *check:* $n=10$, *exact*=0.276, *asymptotic*=0.319

☞ $P(n,n+1)=2^n e^{2-n} n^{1/2}\xi\left(\frac{5}{12}-\frac{7}{12}\xi n^{-1/2}+\frac{515}{144}n^{-1}-\frac{28}{9}\xi n^{-3/2}+\frac{788347}{51840}n^{-}\right.$

- ▷ *check:* $n=10$, *exact*=0.437, *asymptotic*=0.407
- ▷ *check:* $n=20$, *exact*=0.037108, *asymptotic*=0.037245
- ▷ *check:* $n=100$, *exact*=2.617608×10^{-12}, *asymptotic*=2.617596×10^{-12}

References

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[jklp93] S Janson, D E Knuth, T Luczak & B G Pittel: The birth of the giant component *Random Structures and Algorithms*, **4**, 233-358 (1993)
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