Connectivity of random graphs

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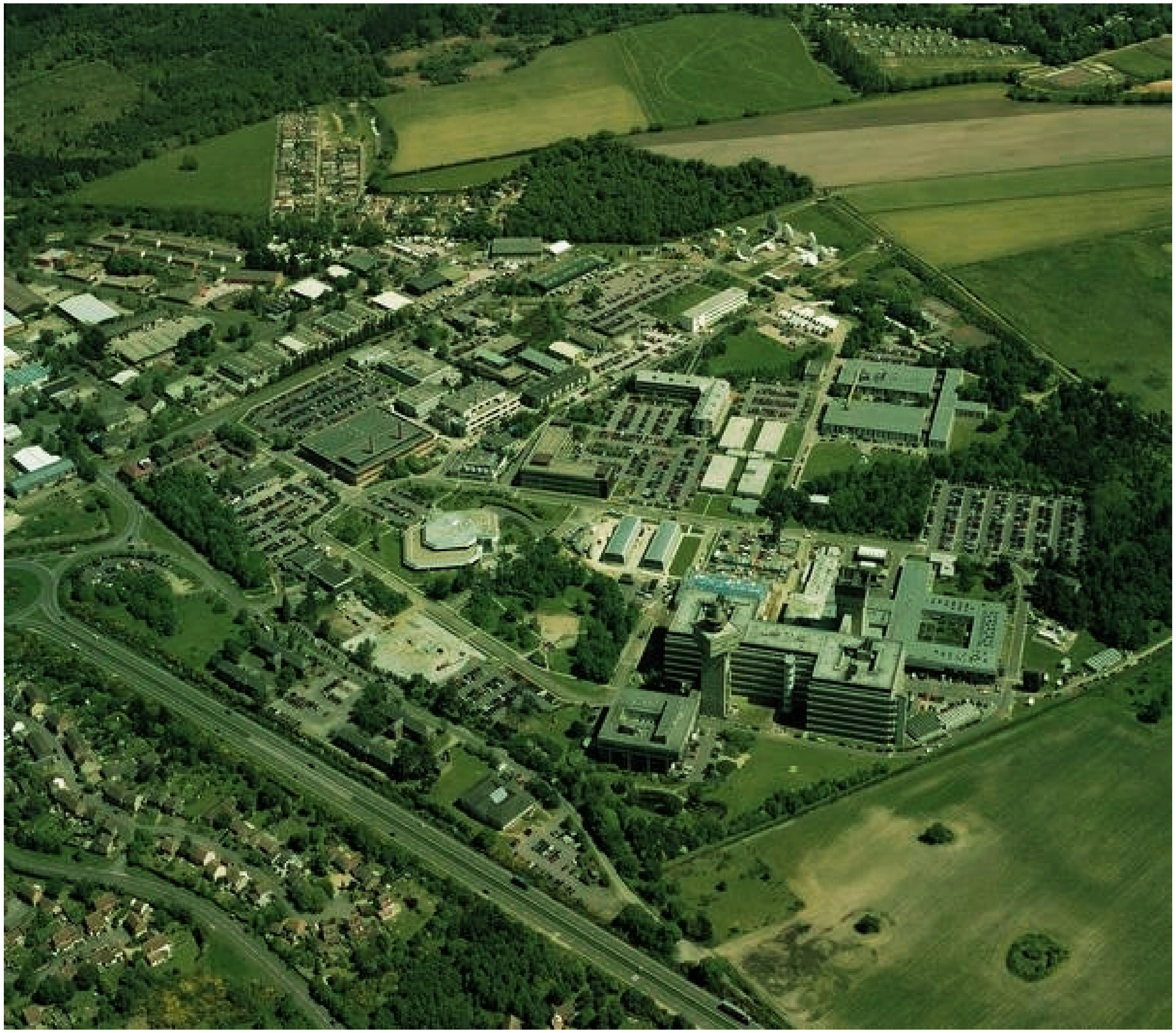
Alan Turing Institute 2004 November 19 1330

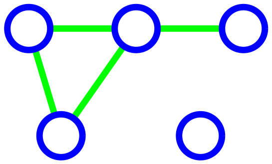
typeset 2004 November 17 10:55 in pdfLaTeX on a linux system

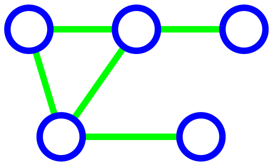
BT Research at Martlesham, Suffolk

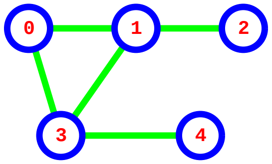


- Cambridge-Ipswich high-tech corridor
- 2000 technologists
- 15 companies
- UCL, Univ of Essex

Graphs

- (simple unlabelled undirected) graph: 

- (simple unlabelled undirected) connected graph: 

- (simple undirected) labelled graph: 

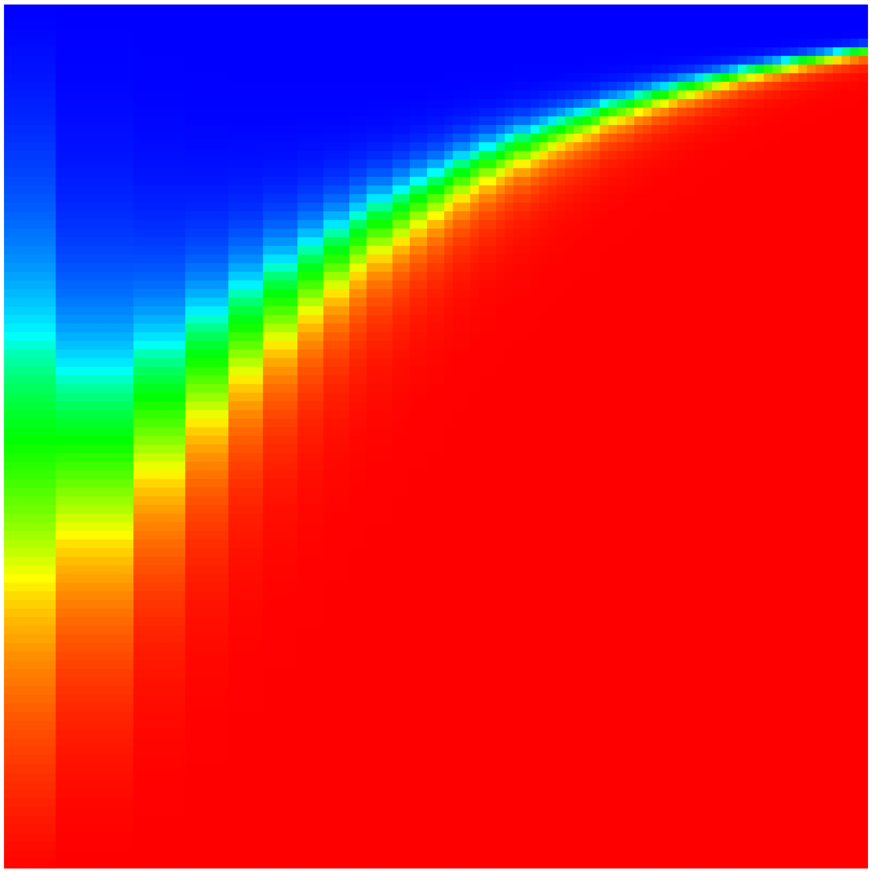
The Bernoulli random graph model $G\{n, p\}$

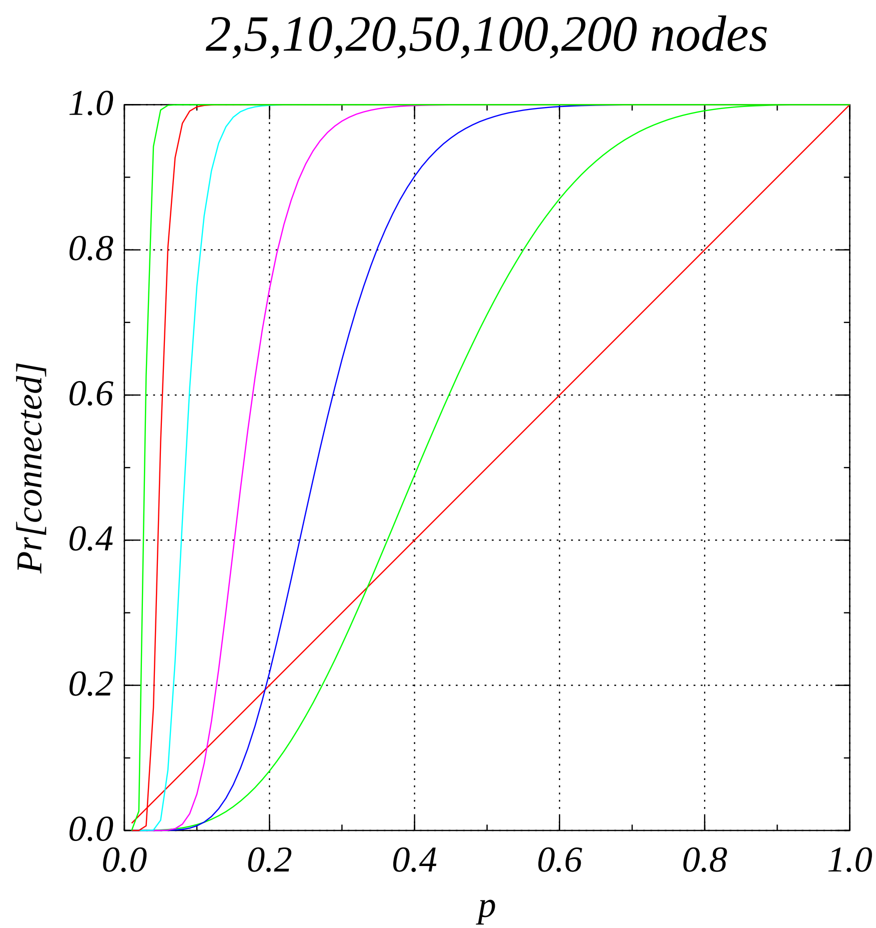
- let G be a graph of n nodes
- let $p = 1 - q$ be the probability that each possible edge exists
- edge events are independent
- let $P(n)$ be the probability that $G\{n, p\}$ is connected
- then $P(1) = 1$ and $P(n) = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} P(k) q^{k(n-k)}$ for $n = 2, 3, 4, \ldots$.

$$
\begin{aligned}
P(1) &= 1 \\
P(2) &= 1 - q \\
P(3) &= (2\,q + 1)\,(q - 1)^2 \\
P(4) &= \left(6\,q^3 + 6\,q^2 + 3\,q + 1\right)(1 - q)^3 \\
P(5) &= \left(24\,q^6 + 36\,q^5 + 30\,q^4 + 20\,q^3 + 10\,q^2 + 4\,q + 1\right)(q - 1)^4
\end{aligned}
$$

- as $n \to \infty$, we have $P(n) \to 1 - nq^{n-1}$.

Connectivity for the Bernoulli model





x-axis: $\log(n = \text{number of nodes}), n = 2, \ldots, 100$
y-axis: p, 0 at top, 1 at bottom
blue=0 red=1

Probability of connectivity - the $G(n,m)$ model

- problem: compute the numbers of connected labelled graphs with n nodes and $m=n-1,n,n+1,n+2,\dots$ edges
  - ▷ *with this information, we can compute the probability of a randomly chosen labelled graph being connected*
- compute large-n asymptotics for these quantities, where the number of edges is only slightly larger than the number of nodes
- I did some exact numerical calculations to try to establish the dominant asymptotics
- I then looked at some earlier papers and found that the required theory to compute exact asymptotics *is* known
- I computed the exact asymptotics and got perfect agreement with my exact numerical data

The inspirational paper [fss04]

- Philippe Flajolet, Bruno Salvy and Gilles Schaeffer: *Airy Phenomena and Analytic Combinatorics of Connected Graphs*

- The claim: the number $C(n, n+k)$ of labelled (étiquetés) connected graphs with n nodes and excess (edges-nodes) $= k \geqslant 2$ is

$$A_k(1)\sqrt{\pi}\left(\frac{n}{e}\right)^n\left(\frac{n}{2}\right)^{\frac{3k-1}{2}}\left[\frac{1}{\Gamma(3k/2)}+\frac{A_k'(1)/A_k(1)-k}{\Gamma((3k-1)/2)}\sqrt{\frac{2}{n}}+\mathcal{O}\left(\frac{1}{n}\right)\right]$$

-

k	1	2	3	4	5	6	7
$A_k(1)$	5/24	5/16	1105/1152	565/128	82825/3072	19675/96	1282031525/688128
$A_k'(1)$	19/24	65/48	1945/384	21295/768	603965/3072	10454075/6144	1705122725/98304

- Airy in Playford:
`www.ast.cam.ac.uk/~ipswich/History/Airys_Country_Retreat.htm`

Some problems with the paper

- I did some comparisons with exact counts for up to $n=1000$ nodes and for excess $k=2,3,\ldots,8$
- The exact data was computed from the generating functions
- The fit was very bad
- This formula was found to fit the data much better for $k=2$:

$$A_k(1)\sqrt{\pi}n^n\left(\frac{n}{2}\right)^{\frac{3k-1}{2}}\left[\frac{1}{\Gamma(3k/2)}-\frac{A_k'(1)/A_k(1)-k}{\Gamma((3k-1)/2)}\sqrt{\frac{2}{n}}+\mathcal{O}\left(\frac{1}{n}\right)\right]$$

- Also, on pages 4 and 24, I think S should have the expansion $1-(5/4)\alpha+(15/4)\alpha^2+\ldots$

Asymptotic expansion of $C(n,n+k)/n^{n+\frac{3k-1}{2}}$

$\xi \equiv \sqrt{2\pi}$ green: from [bcm90] red: from [fss04] (with removal of factor e)

k	type	$[n^0]$	$[n^{-1/2}]$	$[n^{-1}]$	$[n^{-3/2}]$
-1	tree	1	0	0	0
0	unicycle	$\xi\frac{1}{4}$			
1	bicycle	$\frac{5}{24}$			
2	tricycle	$\xi\frac{5}{256}$ $\xi\frac{5}{256}$	$-\frac{35}{144}$		
3	quadricycle	$\frac{221}{1512}$ $\frac{221}{24192}$	$-\sqrt{\pi}\frac{35}{96}$		
4	pentacycle	$\xi\frac{113}{196608}$			

blue: conjectured by KMB from numerical experiments

k	type	$[n^0]$	$[n^{-1/2}]$	$[n^{-1}]$	$[n^{-3/2}]$	$[n^{-2}]$	$[n^{-5/2}]$
0	unicycle	$\xi\frac{1}{4}$	$-\frac{7}{6}$	$\xi\frac{1}{48}$	$\frac{131}{270}$	$\xi\frac{1}{1152}$	$-\frac{4}{2835}$?
1	bicycle	$\frac{5}{24}$	$-\xi\frac{7}{24}$	$\frac{25}{36}$	$-\xi\frac{7}{288}$	$-\frac{79}{3240}$?	
2	tricycle	$\xi\frac{5}{256}$	$-\frac{35}{144}$	$\xi\frac{1559}{9216}$	$-\frac{55}{144}$		
3	quadricycle	$\frac{221}{24192}$	$-\xi\frac{35}{10706}$				

Counting

- Given a finite set S, how many elements $|S|$ does it have?
- The classical method for computing $|S|$:

$$|S| = \sum_{x \in S} 1$$

Definitions of generating functions

- generating function (gf):

$$\{a_1, a_2, a_3, \ldots\} \leftrightarrow \sum_{k=1}^{\infty} a_k x^k$$

- exponential generating function (egf):

$$\{a_1, a_2, a_3, \ldots\} \leftrightarrow \sum_{k=1}^{\infty} \frac{a_k}{k!} x^k$$

Exponential generating functions

- exponential generating function for all labelled graphs:

$$g(w,z)=\sum_{n=0}^{\infty}(1+w)^{\binom{n}{2}}z^n/n!$$

- exponential generating function for all connected labelled graphs:

$$\begin{aligned} c(w,z) &= \log(g(w,z)) \\ &= z+w\frac{z^2}{2}+(3w^2+w^3)\frac{z^3}{6}+(16w^3+15w^4+6w^5+w^6)\frac{z^4}{4!}+\dots \end{aligned}$$

egfs for labelled graphs [jklp93]

- rooted labelled trees

$$T(z) = z\exp(T(z)) = \sum n^{n-1}\frac{z^n}{n!} = z + \tfrac{2}{2!}z^2 + \tfrac{9}{3!}z^3 + \cdots$$

- unrooted labelled trees

$$W_{-1}(z) = T(z) - T(z)^2/2 = z + \tfrac{1}{2!}z^2 + \tfrac{3}{3!}z^3 + \tfrac{16}{4!}z^4 + \ldots$$

- unicyclic labelled graphs

$$W_0(z) = \frac{1}{2}\log\left[\frac{1}{1-T(z)}\right] - \frac{1}{2}T(z) - \frac{1}{4}T(z)^2 = \tfrac{1}{3!}z^3 + \tfrac{15}{4!}z^4 + \tfrac{222}{5!}z^5 + \tfrac{3660}{6!}z^6 + .$$

- bicyclic labelled graphs

$$W_1(z) = \frac{T(z)^4\big(6 - T(z)\big)}{24\big(1 - T(z)\big)^3} = \tfrac{6}{4!}z^4 + \tfrac{205}{5!}z^5 + \tfrac{5700}{6!}z^6 + \ldots$$

Theory 1

The previous observations can be proved using theory available in [jklp93] and [fgkp95]. I sketch the computations.

- Ramanujan's Q-function is defined for $n = 1, 2, 3, \ldots$:

$$Q(n) \equiv \sum_{k=1}^{\infty} \frac{n^{\underline{k}}}{n^k} = 1 + \frac{n-1}{n} + \frac{(n-1)(n-2)}{n^2} + \ldots$$

- $\sum_{n=1}^{\infty} Q(n) n^{n-1} \frac{z^n}{n!} = -\log(1 - T(z))$
- here T is the egf for rooted labelled trees: $T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n$
- $T(z) = z \exp(T(z))$

Theory 2

- to get the large-n asymptotics of Q, we first consider the related function $R(n)\equiv 1+\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\dots,\ n=1,2,3,\dots$
  - ▷ *we have* $Q(n)+R(n)=n!\,e^n/n^n$
  - ▷ *let* $D(n)=R(n)-Q(n)$
  - ▷ $\sum_{n=1}^{\infty} D(n)n^{n-1}\frac{z^n}{n!}=\log[\frac{(1-T(z))^2}{2(1-ez)}]$
  - ▷ $D(n)\sim\sum_{k=1}^{\infty} c(k)[z^n](T(z)-1)^k$, *where* $c(k)\equiv[\delta^k]\log(\delta^2/2/(1-(1+\delta)e^{-\delta}))$
  - ▷ *this gives* $D(n)\sim\frac{2}{3}+\frac{8}{135}\,n^{-1}-\frac{16}{2835}\,n^{-2}-\frac{32}{8505}\,n^{-3}+\frac{17984}{12629925}\,n^{-4}+\frac{668288}{492567075}\,n^{-5}+O\left(n^{-6}\right)$

- now using $Q(n)=(n!\,e^n/n^n-D(n))/2$, we get
  - ▷ $Q(n)\sim\frac{1}{2}n^{1/2}\sqrt{2\pi}-\frac{1}{3}+\frac{1}{24}\sqrt{2\pi}n^{-1/2}-\frac{4}{135}\,n^{-1}+\frac{1}{576}\sqrt{2\pi}n^{-3/2}+\frac{8}{2835}\,n^{-2}-\frac{139}{103680}\sqrt{2\pi}n^{-5/2}+\frac{16}{8505}\,n^{-3}-\frac{571}{4976640}\sqrt{2\pi}n^{-7/2}-\frac{8992}{12629925}\,n^{-4}+\frac{163879}{418037760}\sqrt{2\pi}n^{-9/2}-\frac{334144}{492567075}\,n^{-5}+\frac{5246819}{150493593600}\sqrt{2\pi}n^{-11/2}+O\left(n^{-6}\right)$

Theory 3

- Let W_k be the egf for connected labelled $(k+1)$-cyclic graphs
  - *for unrooted trees* $W_{-1}(z) = T(z) - T^2(z)/2$, $[z^n]W_{-1}(z) = n^{n-2}$
  - *for unicycles* $W_0(z) = -(\log(1-T(z)) + T(z) + T^2(2)/2)/2$
  - *for bicycles* $W_1(z) = \frac{6T^4(z) - T^5(z)}{24(1-T(z))^3}$
  - *for* $k \geqslant 1$, $W_k(z) = \frac{A_k(T(z))}{(1-T(z))^{3k}}$, *where* A_k *are polynomials computable from results in [jklp93]*

- Knuth and Pittel's tree polynomials $t_n(y)$ ($y \neq 0$) are defined by $(1-T(z))^{-y} = \sum_{n=0}^{\infty} t_n(y) \frac{z^n}{n!}$
  - *we can compute these for* $y > 0$ *from*
    $t_n(1) = 1; \quad t_n(2) = n^n(1+Q(n)); t_n(y+2) = n\, t_n(y)/y + t_n(y+1)$
- thanks to this recurrence, the asymptotics for t_n follow from the known asymptotics of Q

Theory 4

Let $\xi=\sqrt{2\pi}$. All results agree with numerical estimates on this page.

- the number of connected unicycles is $C(n,n)=n![z^n]W_0(z)=\frac{1}{2}Q(n)n^{n-1}+3/2+t_n(-1)-t_n(-2)/4$
  - ▷ $\frac{C(n,n)}{n^n}\sim\frac{1}{4}\,\xi n^{-1/2}-\frac{7}{6}\,n^{-1}+\frac{1}{48}\,\xi n^{-3/2}+\frac{131}{270}\,n^{-2}+\frac{1}{1152}\,\xi n^{-5/2}+\frac{4}{2835}\,n^{-3}-\frac{139}{207360}\,\xi n^{-7/2}+\frac{8}{8505}\,n^{-4}-\frac{571}{9953280}\,\xi\left(n^{-1}\right)^{9/2}-\frac{4496}{12629925}\,n^{-5}+\frac{163879}{836075520}\,\xi n^{-11/2}+O\left(n^{-6}\right)$
- the number of connected bicycles is $C(n,n+1)=n![z^n]W_1(z)=\frac{5}{24}t_n(3)-\frac{19}{24}t_n(2)+\frac{13}{12}t_n(1)-\frac{7}{12}t_n(0)+\frac{1}{24}t_n(-1)+\frac{1}{24}t_n(-2)$
  - ▷ $\frac{C(n,n+1)}{n^n}\sim\frac{5}{24}\,n-\frac{7}{24}\,\xi n^{1/2}+\frac{25}{36}-\frac{7}{288}\,\xi n^{-1/2}-\frac{79}{3240}\,n^{-1}-\frac{7}{6912}\,\xi n^{-3/2}-\frac{413}{4860}\,n^{-2}+\frac{973}{1244160}\,\xi n^{-5/2}-\frac{4}{3645}\,n^{-3}+\frac{3997}{59719680}\,\xi n^{-7/2}+\frac{2248}{5412825}\,n^{-4}-\frac{163879}{716636160}\,\xi n^{-9/2}+\frac{83536}{211100175}\,n^{-5}-\frac{5246819}{257989017600}\,\xi n^{-11/2}+O\left(n^{-6}\right)$
- similarly, for the number of connected tricycles we get
  - ▷ $\frac{C(n,n+2)}{n^n}\sim\frac{5}{256}\,\xi n^{5/2}-\frac{35}{144}\,n^2+\frac{1559}{9216}\,\xi n^{3/2}-\frac{55}{144}\,n+\frac{33055}{221184}\,\xi n^{1/2}-\frac{41971}{136080}+\frac{31357}{2654208}\,\xi n^{-1/2}+\frac{1129}{81648}\,n^{-1}+O\left(n^{-3/2}\right)$

Probability of connectivity 1

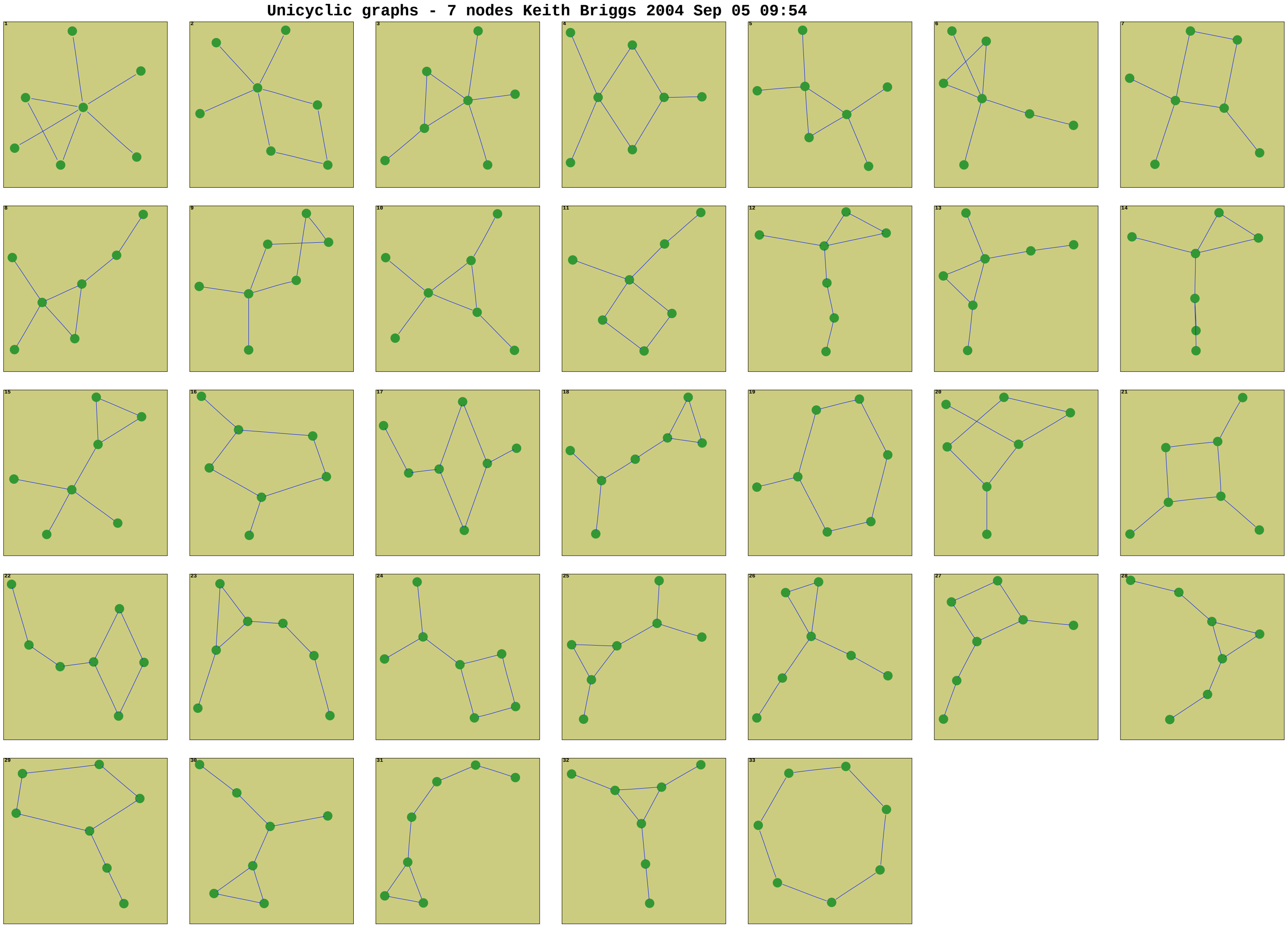
- we now have all the results needed to calculate the asymptotic probability $P(n,n+k)$ that a randomly chosen graph with n nodes and $n+k$ edges is connected (for $n\to\infty$ and small fixed k)

- the total number of graphs is $g(n,n+k)\equiv\binom{\binom{n}{2}}{n+k}$. This can be asymptotically expanded:
  - $\frac{g(n,n-1)}{\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{-3/2}} \sim 1+\frac{7}{4}n^{-1}+\frac{259}{96}n^{-2}+\frac{22393}{5760}n^{-3}+\frac{54359}{10240}n^{-4}+\frac{52279961}{7741440}n^{-5}+\frac{777755299}{103219200}n^{-6}+O\left(n^{-7}\right)$
  - $\frac{g(n,n+0)}{\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{-1/2}} \sim \frac{1}{2}-\frac{5}{8}n^{-1}-\frac{53}{192}n^{-2}-\frac{4067}{11520}n^{-3}-\frac{9817}{20480}n^{-4}-\frac{10813867}{15482880}n^{-5}-\frac{217565701}{206438400}n^{-6}-\frac{11591924473}{7431782400}n^{-7}+O\left(n^{-8}\right)$
  - $\frac{g(n,n+1)}{\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{3/2}} \sim \frac{1}{4}-\frac{21}{16}n^{-1}+\frac{811}{384}n^{-2}-\frac{43187}{23040}n^{-3}+\frac{159571}{73728}n^{-4}-\frac{55568731}{30965760}n^{-5}+\frac{2867716177}{1238630400}n^{-6}-\frac{3215346127}{2123366400}n^{-7}+\frac{1317595356557}{475634073600}n^{-8}+O\left(n^{-9}\right)$
  - $\dots$
  - $g(n,n+k)\sim\sqrt{\frac{2}{\pi}}e^{n-2}\left(\frac{n}{2}\right)^n n^{k-1/2}\left(2^{-k-1}+O(n^{-1})\right)$

Probability of connectivity 2

- $\frac{P(n,n-1)}{2^n e^{2-n} n^{-1/2} \xi} \sim \frac{1}{2} - \frac{7}{8} n^{-1} + \frac{35}{192} n^{-2} + \frac{1127}{11520} n^{-3} + \frac{5189}{61440} n^{-4} + \frac{457915}{3096576} n^{-5} + \frac{570281371}{1857945600} n^{-6} + \frac{291736667}{495452160} n^{-7} + O\left(n^{-8}\right)$
  - ▷ *check:* $n = 10$, *exact*=0.1128460393, *asymptotic*=0.1128460359
- $\frac{P(n,n+0)}{2^n e^{2-n} \xi} \sim \frac{1}{4}\xi - \frac{7}{6} n^{-1/2} + \frac{1}{3}\xi n^{-1} - \frac{1051}{1080} n^{-3/2} + \frac{5}{9}\xi n^{-2} + O\left(n^{-3}\right)$
  - ▷ *check:* $n = 10$, *exact*=0.276, *asymptotic*=0.319
- $\frac{P(n,n+1)}{2^n e^{2-n} n^{1/2} \xi} \sim \frac{5}{12} - \frac{7}{12}\xi n^{-1/2} + \frac{515}{144} n^{-1} - \frac{28}{9}\xi n^{-3/2} + \frac{788347}{51840} n^{-2} - \frac{308}{27}\xi n^{-5/2} + O\left(n^{-3}\right)$
  - ▷ *check:* $n = 10$, *exact*=0.437, *asymptotic*=0.407
  - ▷ *check:* $n = 20$, *exact*=0.037108, *asymptotic*=0.037245
  - ▷ *check:* $n = 100$, *exact*=2.617608×10^{-12}, *asymptotic*=2.617596×10^{-12}

Example of unlabelled case - unicycles for $n = 7$



The unlabelled case - unicycles

- Christian Bower's idea: A connected unicyclic graph is an undirected cycle of 3 or more rooted trees. Start with a single undirected cycle (or polygon) graph. It must have at least 3 nodes. Hanging from each node in the cycle is a tree (a tree is of course a connected acyclic graph). The node where the tree intersects the cycle is the root, thus it is (combinatorially) a rooted tree.
  - ▷ *A000081 is undirected cycles of exactly 1 rooted tree*
  - ▷ *A001429 is undirected cycles of 3 or more rooted trees*
  - ▷ *A027852 is undirected cycles of exactly 2 rooted trees*
  - ▷ *A068051 is undirected cycles of 1 or more rooted trees*
- this gives formulae which should be amenable to analysis, but let's first do some brute-force numerics to get a feel for the behaviour
- I calculated the counts up to $n = 20000$ nodes, and tried to guess the form of the asymptotic expansion and then the values of the coefficients

Transforms

- A027852 is undirected cycles of exactly 2 rooted trees

$$A_{27852}(x) \;=\; \frac{1}{2}\left(A_{81}(x)^2+A_{81}(x^2)\right)$$

- A068051 is undirected cycles of 1 or more rooted trees

$$A_{68051}(x) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{\phi(k)}{k}\left(\log(1-A_{81}(x^k))\right)+$$
$$\frac{2A_{81}(x)+A_{81}(x)^2+A_{81}(x^2)}{4(1-A_{81}(x^2))}$$

- A001429 is undirected cycles of 3 or more rooted trees

$$A_{1429}(x) \;=\; A_{68051}(x)-A_{27852}(x)-A_{81}(x)$$

Asymptotics of A81 (unlabelled rooted trees) [fin03]

- $d = 2.9557652856519949747148175241231\ldots$
- $\frac{A81(n)}{d^n} n^{3/2} \sim$

$$
\begin{aligned}
&0.43992401257102530404090339 14 \quad + \\
&0.44169901840103993692628088 77\, n^{-1} + \\
&0.22169280597203680622202567 92\, n^{-2} + \\
&0.86765549082880896333845501 25\, n^{-3} + \\
&0.62521976227219446953559183 18\, n^{-4} + \\
&32.3253941706451396137450650501\, n^{-5} + \ldots
\end{aligned}
$$

Asymptotics for unlabelled unicycles

- I get that the number $c(n,n)$ of connected unlabelled unicyclic graphs behaves like

$$\left(\frac{c(n,n)}{d^n}-\frac{1}{4^n}\right)n^{3/2} \sim$$
$$-0.4466410059+0.44311055235n^{-1}+0.91158865326n^{-2}+O(n^{-3})$$

- the coefficients are poorly determined (unlike in a similar analysis of the labelled case)

References

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[jklp93] S Janson, D E Knuth, T Luczak & B G Pittel: The birth of the giant component *Random Structures and Algorithms*, **4**, 233-358 (1993)
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