# Connectivity of random graphs

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CABDyN seminar, Saïd Business School 2004 October 12

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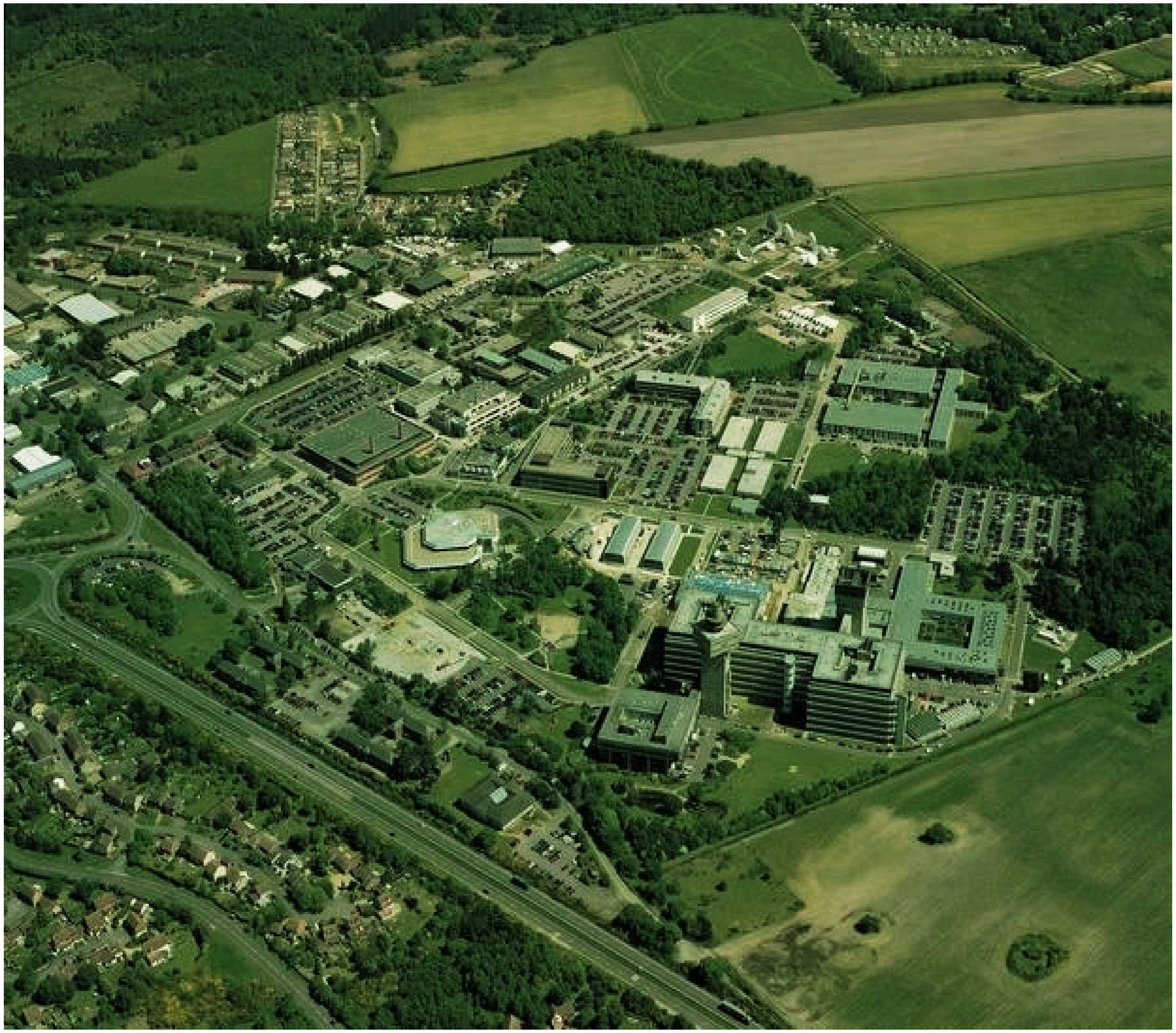
# BT Research at Martlesham, Suffolk



- Cambridge-Ipswich high-tech corridor
- 2000 technologists
- 15 companies
- UCL, Univ of Essex

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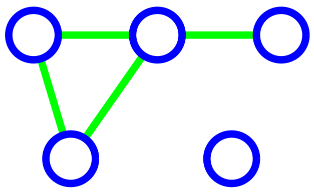
- (simple unlabelled undirected) graph: 

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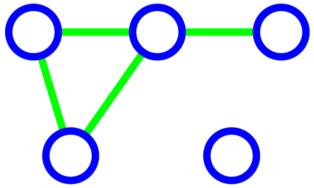
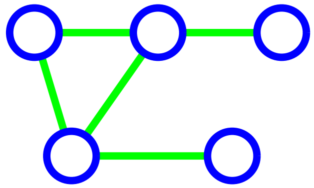
- (simple unlabelled undirected) graph: 
- (simple unlabelled undirected) connected graph: 

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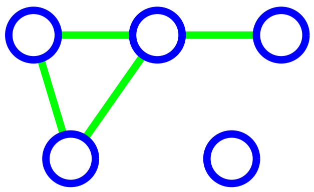

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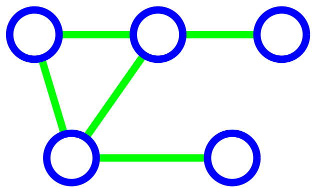
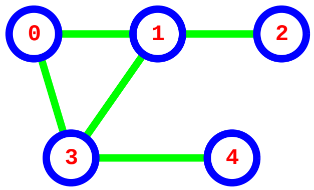


- (simple undirected) labelled graph:



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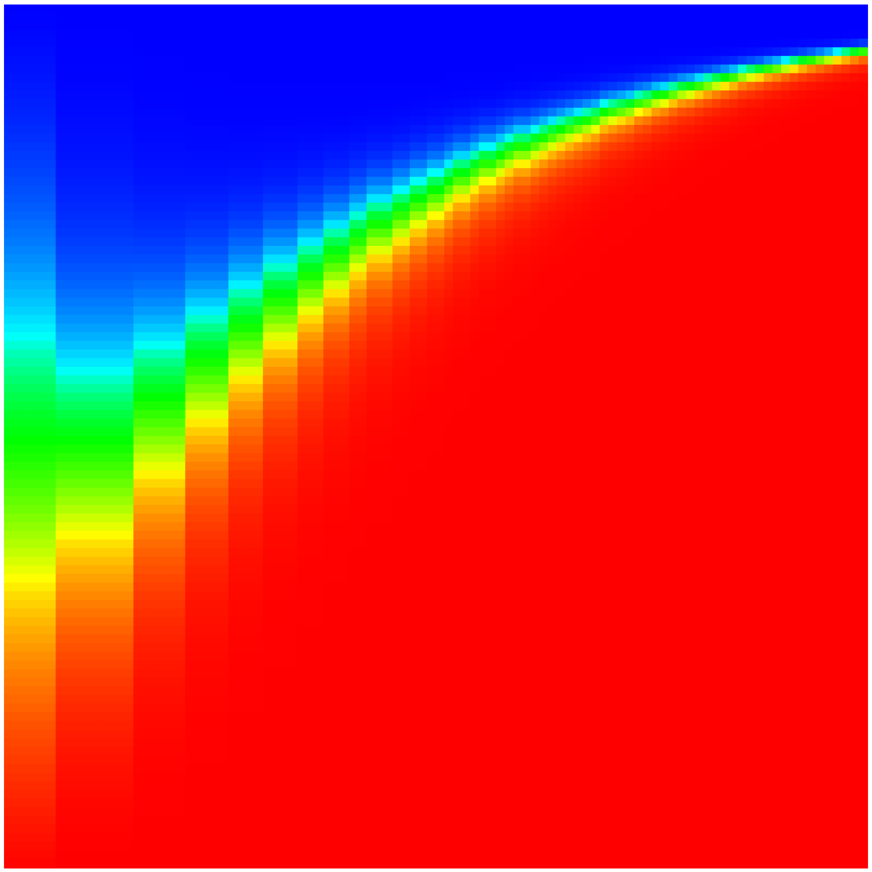
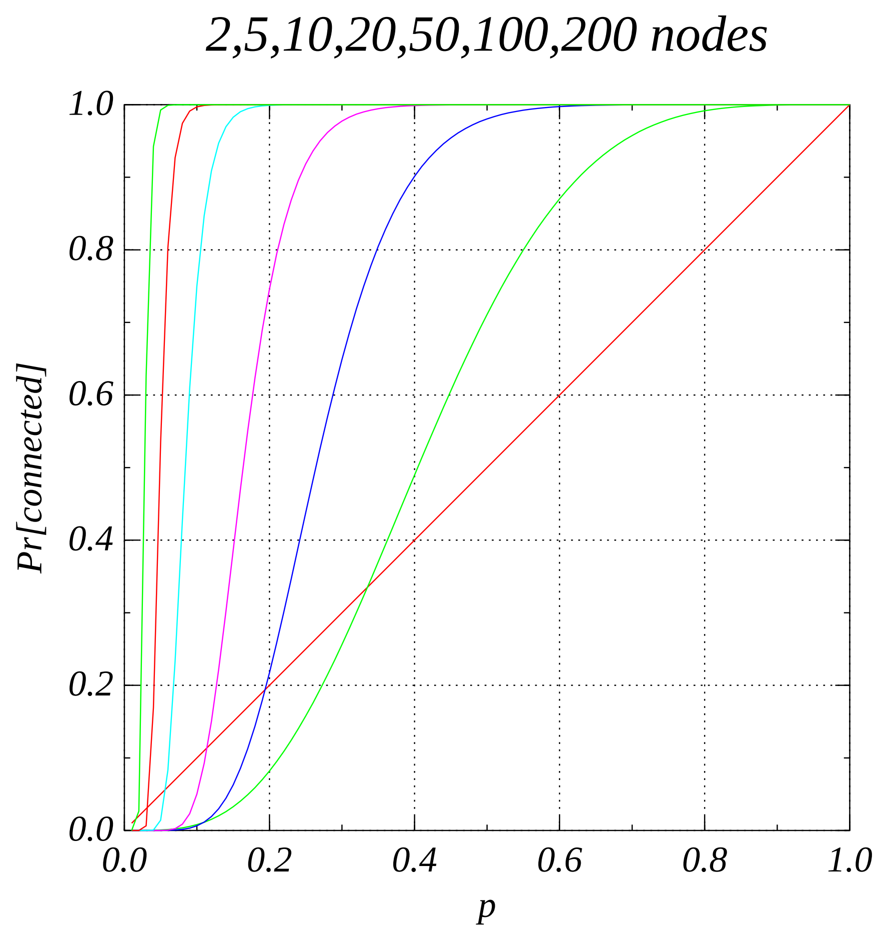
- as $n\to\infty$, we have $P(n,p)\to 1-nq^{n-1}$

# Connectivity for the Bernoulli model



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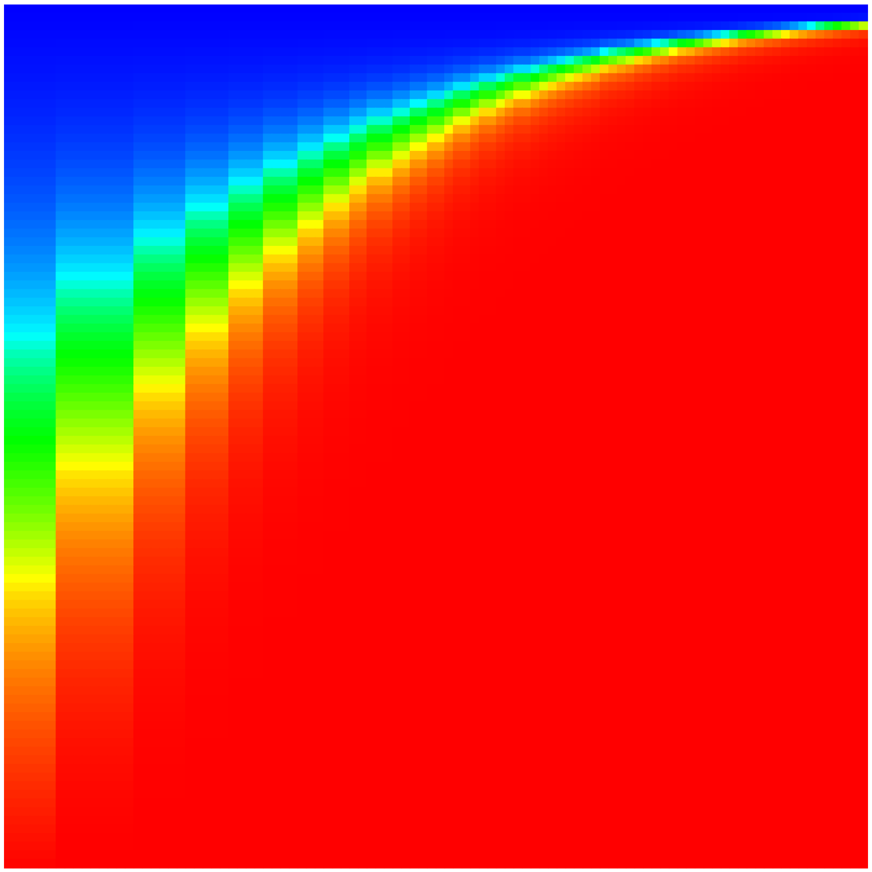
blue=0 red=1

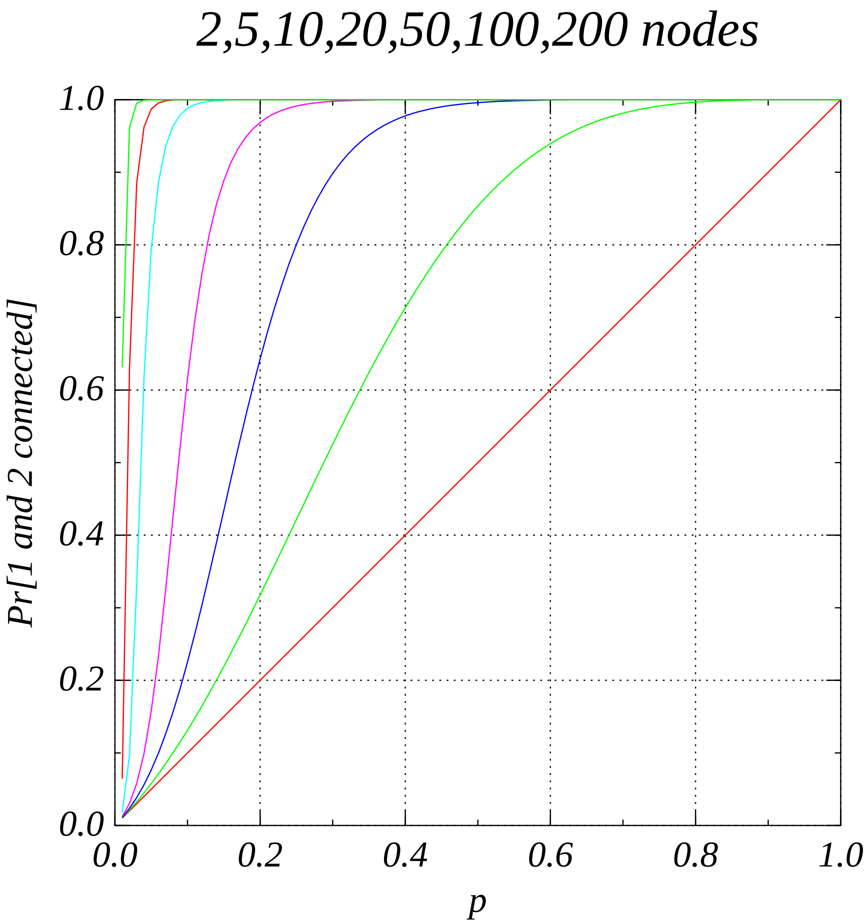
# More on the Bernoulli random graph model

- the probability that two *pre-specified* nodes (not randomly chosen) are connected is different. If we call this $R(n,p)$, then we have $R(1,p)=1$ and:

$$R(n,p)=1-\sum_{k=1}^{n-1}\binom{n-2}{k-1}P(k,p)q^{k(n-k)}, \quad n=2,3,4,\dots$$

# $\Pr[1 \text{ and } 2 \text{ connected}]$ for the Bernoulli model



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- I have computed the accurate asymptotics and have checked the results against exact numerical data
  - a sketch of the ideas involved follows; full details are available on request

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# Some known exponential generating functions

- exponential generating function enumerating labelled graphs ($[z^n]$: number with $n$ nodes; $[w^m]$: number with $m$ edges):

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- exponential generating function enumerating connected labelled graphs:

$$\begin{aligned} c(w,z) &= \log(g(w,z)) \\ &= z + w\frac{z^2}{2} + (3w^2+w^3)\frac{z^3}{6} + (16w^3+15w^4+6w^5+w^6)\frac{z^4}{4!} + \dots \end{aligned}$$

# egfs for labelled graphs [jklp93]

- rooted labelled trees

$$T(z) = z\exp(T(z)) = \sum_{n\geqslant 1} n^{n-1}\frac{z^n}{n!} = z + \tfrac{2}{2!}z^2 + \tfrac{9}{3!}z^3 + \cdots$$

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- bicyclic labelled graphs

$$W_1(z) = \frac{T(z)^4\big(6-T(z)\big)}{24\big(1-T(z)\big)^3} = \tfrac{6}{4!}z^4+\tfrac{205}{5!}z^5+\tfrac{5700}{6!}z^6+\dots$$

# Introduction to asymptotic expansions

- Stirling:

$$\Gamma(n) \sim \left(\frac{2\pi}{n}\right)^{1/2} \left(\frac{n}{e}\right)^{n} \left[1+\frac{1}{12}n^{-1}+\frac{1}{288}n^{-2}-\frac{139}{51840}n^{-3}+\dots\right]$$

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- cf. the Taylor series - 3 terms give an absolute error $>5$
- asymptotic expansion diverges for all $n$!

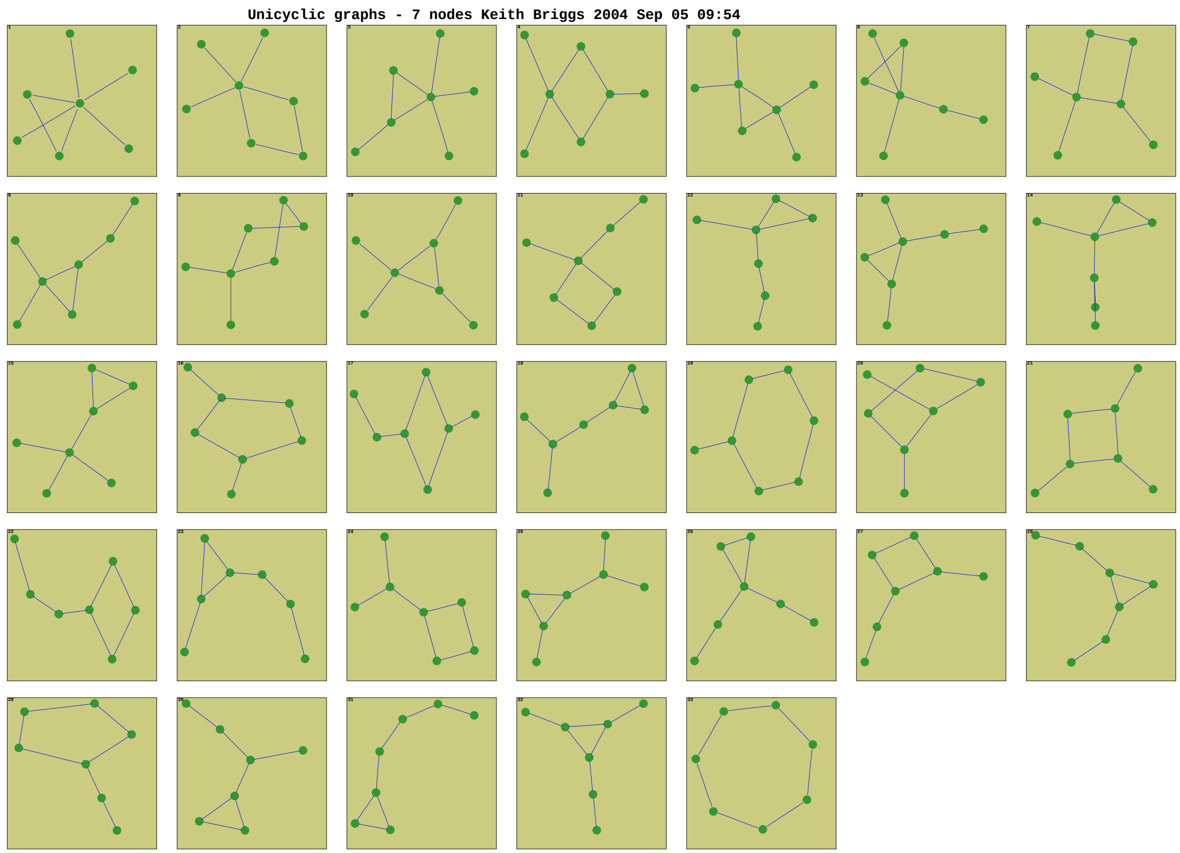
# Asymptotic expansion of $P(n,n+k)$

| $k$ | type | | $[n^0]$ | $[n^{-1/2}]$ | $[n^{-1}]$ | $[n^{-3/2}]$ |
|---|---|---|---|---|---|---|
| $-1$ | tree | $\frac{P(n,n-1)}{2^n e^{2-n} n^{-1/2} \xi}$ | $\frac{1}{2}$ | $0$ | $-\frac{7}{8}$ | $0$ |
| $0$ | unicycle | $\frac{P(n,n+0)}{2^n e^{2-n} \xi}$ | $\frac{1}{4}\xi$ | $-\frac{7}{6}$ | $\frac{1}{3}\xi$ | $-\frac{1051}{1080}$ |
| $1$ | bicycle | $\frac{P(n,n+1)}{2^n e^{2-n} n^{1/2} \xi}$ | $\frac{5}{12}$ | $-\frac{7}{12}\xi$ | $\frac{515}{144}$ | $-\frac{28}{9}\xi$ |

$\xi \equiv \sqrt{2\pi}$

# Unlabelled graphs

- much less is known about the unlabelled case
- the difficulties arise in distinguishing isomorphic graphs



# Asymptotics for unlabelled unicycles

- I get that the number $c(n,n)$ of connected unlabelled unicyclic graphs behaves like

$$\left(\frac{c(n,n)}{d^n}-\frac{1}{4^n}\right)n^{3/2} \sim$$

$$-0.4466410059+0.44311055235n^{-1}+0.91158865326n^{-2}+O(n^{-3})$$

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Notation:

- ▷ *pdf=probability density function*
- ▷ *cdf=cumulative distribution function*
- ▷ *the notation is sloppy in not distinguishing a RV $X$ and its values $x$*
- ▷ $[\![x]\!]$ *is the indicator function: $1$ if $x$ is true, else $0$*

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  - the pdf of the minimum interval is $f_1(d) = n\lambda e^{-n\lambda d}$
  - the expectation of the minimum interval is $\mathbb{E}[d_{(1)}] = 1/(2\lambda)$, so is half the expectation of the internode distance

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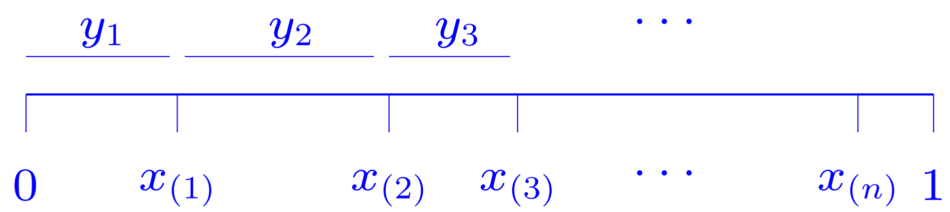
  - the expectation of the minimum interval is $\mathbb{E}[d_{(1)}] = 1/(2\lambda)$, so is half the expectation of the internode distance
  - the probability of full connectivity for the $n$ nodes is thus approximately (i.e. ignoring correlation and edge effects) $F_{n-1}(\rho) = (1-e^{-\lambda\rho})^{n-1}$

# Exact theory for the fixed-$n$ 1d model

- Let $y_k = x_{(k)} - x_{(k-1)}$ be the gaps ($k = 2, \ldots, n$), with $y_1 = x_{(1)}$




- their joint pdf is (for $1 \leqslant m \leqslant n$ and $\sum_{i=1}^{m} y_i \leqslant 1$)

$$f(y_1, y_2, \ldots, y_m) = \frac{n!}{(n-m)!}\left(1 - \sum_{i=1}^{m} y_i\right)^{n-m}$$

- if $c_i$ are constants such that $\sum_{i=1}^{m} c_i \leqslant 1$, then by integrating the pdf we obtain

$$\Pr\left[y_1 > c_1, y_2 > c_2, \ldots\right] = \left(1 - \sum_{i=1}^{m} c_i\right)^{n-1}$$

- Boole's law for the probability of at least one event $A_i$ of $n$ events $A_1, A_2, \ldots, A_n$ occurring is

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_i \Pr\left[A_i\right] - \sum\sum_{i<j} \Pr\left[A_i A_j\right] + \cdots + (-1)^{n-1} \Pr\left[A_1 A_2 \ldots A_n\right]$$

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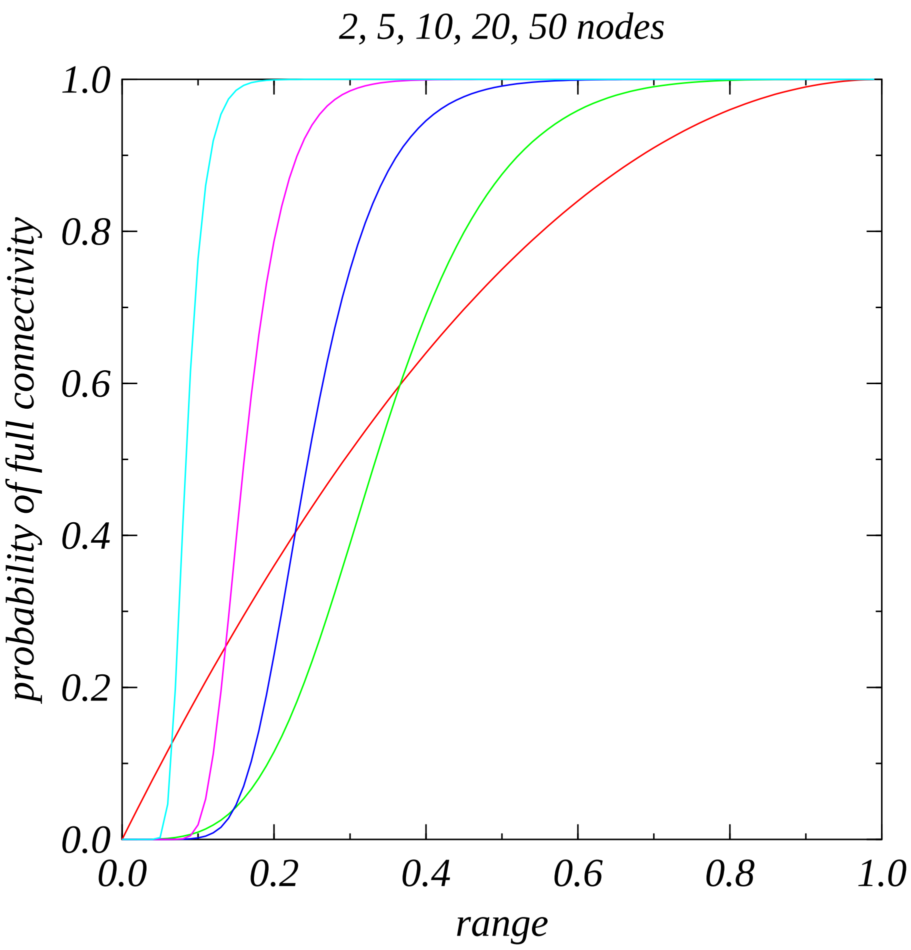
- note that for $\rho > 1/2$, this is exactly $1-(n-1)(1-\rho)^n$

# Probability of connectivity for the fixed-$n$ 1d model




2 5 10 20 50

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- the variance of $d$ is $(4-\pi)/(4\pi\lambda)$
- the probability of a node being isolated (i.e. having no neighbour within range $\rho$) is $e^{-\pi\lambda\rho^2}$

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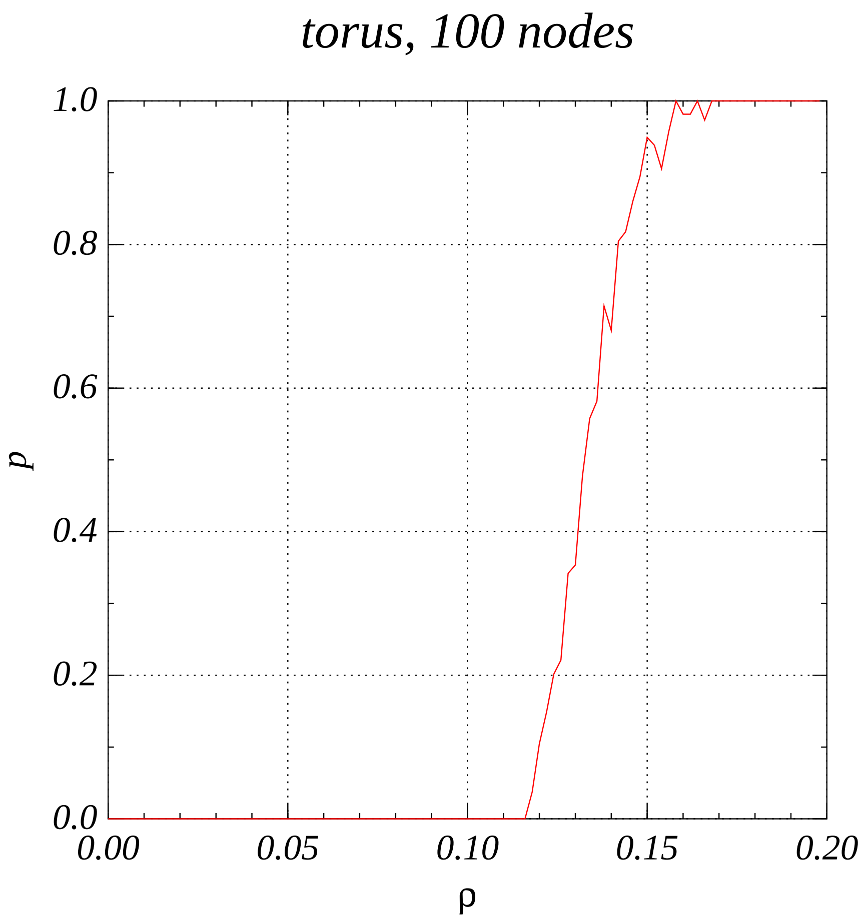
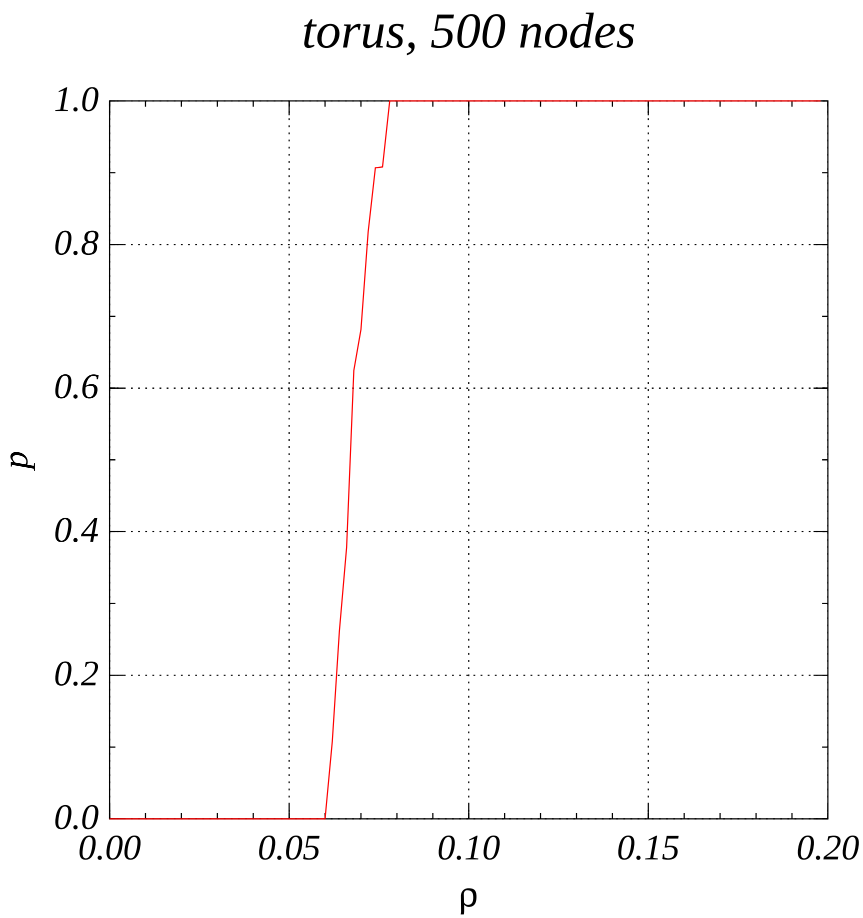
- the blue curve is the asymptotic probability of the whole region $R$ being covered

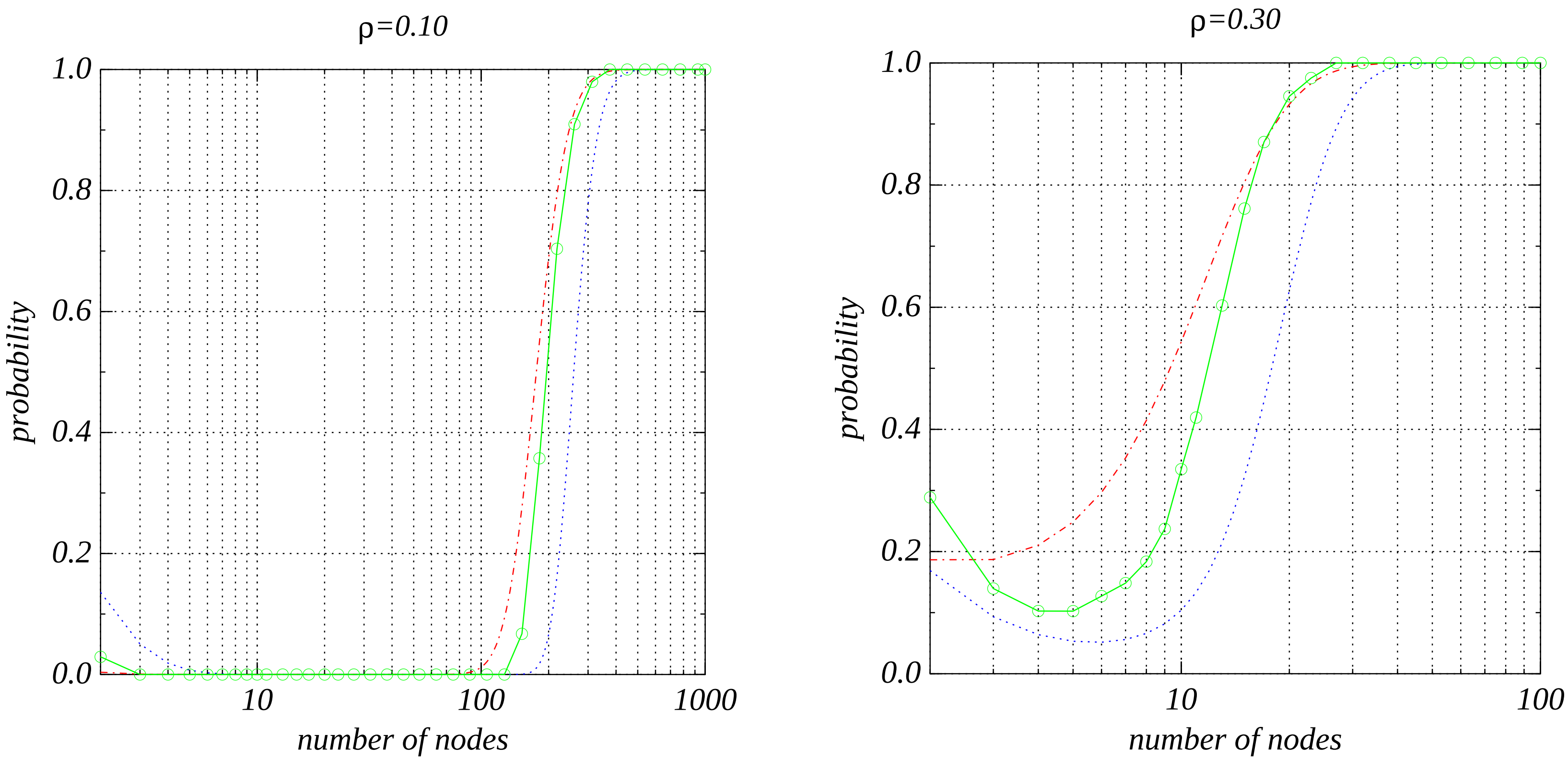
# The phase transition

- the critical radius is $O((\log(n)/(\pi n))^{1/2})$
- for $n = 100$, we estimate $\rho = 0.121$; for $n = 500$, $\rho = 0.0629$



# Simulation results - torus, $\rho = 0.1, 0.3$




$\left[1 - e^{-\pi\lambda\rho^2}\right]^n$ simulation asymptotic

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