Discrete Green's functions on graphs

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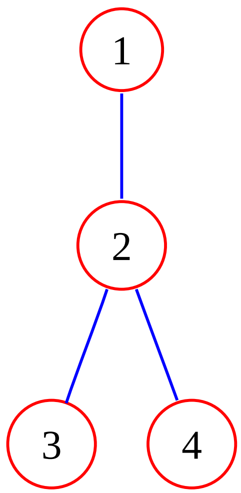
Adjacency matrix

- ★ Let Γ be an arbitrary graph with n nodes
- ★ Let A be the adjacency matrix of Γ

Example:



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Graph eigenvalues

- Let Γ be an arbitrary graph with n nodes
- Let A be the adjacency matrix of Γ
- Let D be the diagonal matrix with D_{xx} the degree of node x
- Let $L \equiv D - A$ be the (combinatorial) Laplacian matrix
- The *normalized Laplacian*, $\mathcal{L} = D^{-1/2} L D^{-1/2}$, is

$$\mathcal{L}(x,y) = \begin{cases} 1, & \text{if } x = y \\ -1/\sqrt{d_x d_y}, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

- The *discrete Laplace operator* Δ is

$$\Delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ -1/d_x, & \text{if } x \sim y \\ 0, & \text{otherwise.} \end{cases}$$

Subgraphs and boundaries

- ⋆ Let S be a subgraph
- ⋆ The boundary is $\partial S = \{y \notin S \text{ s.t. } \exists x \in S, x \sim y\}$
- ⋆ We define the *Dirichlet* versions of L_S, $\mathcal{L}_S$, and Δ_S as the results of deleting the rows and columns corresponding to $\Gamma \backslash S$ from L, $\mathcal{L}$, and Δ, respectively.
- ⋆ Then $S \subsetneq V$, Δ_S, L_S and $\mathcal{L}_S$ are invertible
- ⋆ The *Green's function* G and *normalized Green's function* $\mathcal{G}$ are determined by

$$\Delta_S \, G = G \, \Delta_S = I_S$$

$$\mathcal{L}_S \, \mathcal{G} = \mathcal{G} \, \mathcal{L}_S = I_S$$

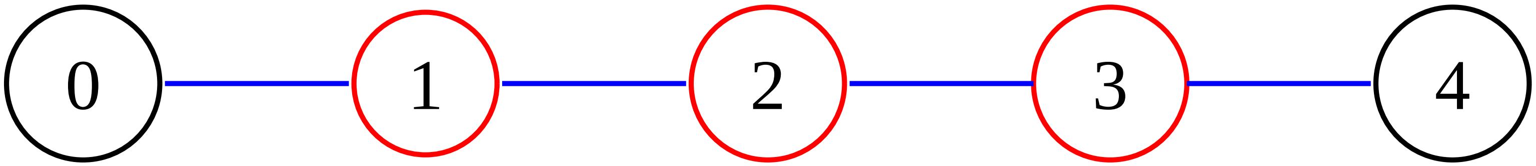
Random walks

★ Let $P(x,y)$ be the transition probability matrix for random walk on S with absorbing states $V \setminus S$, where the probability $P(x,y)$ of moving to state y from state x is $1/d_x$ if $x \sim y$ and 0 otherwise. Then $\Delta_S = I - P$, and $(I-P)^{-1} = I + P + P^2 + \cdots$ gives

$$G(x,y) = \sum_n P_n(x,y),$$

where $P_n(x,y)$ is the n-step transition probability matrix

Example - line graph



- $\Delta f(x) = f(x) - f(x+1)/2 - f(x-1)/2$
- example: 5 nodes, S is the 3 interior nodes
- $2G = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$
- eigenvalues $1 - \cos(\pi/4), 1 - \cos(2\pi/4), 1 - \cos(3\pi/4)$

Hitting times

- The *hitting time* $Q(x, y)$ of a simple random walk starting at vertex x with target vertex y is the expected number of steps to reach vertex y for the first time by starting at x and at each step moving to any neighbour of x with equal probability.

- $$Q(x, y) = \frac{vol(\Gamma)}{d_y}\,\mathcal{G}(y, y) - \frac{vol(\Gamma)}{\sqrt{d_x d_y}}\,\mathcal{G}(x, y).$$

- volume is sum of vertex degrees

Diffusion

- Eigenvalues λ_i and eigenfunctions ϕ_i
- heat kernel

$$H_t(x, y) = \sum_i \exp(-\lambda_i t)\ \phi_i(x)\ \phi_i(y)$$

- satisfies

$$\frac{d}{dt} f = -\mathcal{L}_S\ f$$

- $H_t = \exp(-t\mathcal{L}_S)$
- this solves the diffusion problem - is this useful for virus spreading and gossip algorithms?

References

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