Some quick notes on exponential random graphs

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Why?

- ★ we use many random graph models in network applications. . .
- ★ but rarely specify the statistical ensemble precisely
- ★ so even the averages we compute are suspect
- ★ and even the famous Barabási-Albert scale-free model has known problems
- ★ it's hard to model degree correlations

> we need a unified, rigorous framework

Exponential random graphs

- fix a number of nodes n
- consider the set $G(n)$ of all graphs on n nodes
- we will assign to each $g \in G(n)$ a probability $P(g)$
- let $x = \{x_1, x_2, \ldots\}$ be a set of functions on $G(n)$ representing properties we are interested in, for example
  - $x_1(g)$=*number of edges*
  - $x_2(g)$=*number of nodes of degree 3*
  - $x_3(g)$=*number of triangles*
- we then assign the probabilities P by

$$P_\beta(g) = \frac{1}{Z(\beta)} \exp\left(-\beta_1 x_1 - \beta_2 x_2 - \ldots\right)$$

where $Z(\beta) = \sum_{G(n)} P_\beta(g)$

Statistical mechanics

- Hamiltonian $H(g)=\sum_i \beta_i x_i(g)$
- free energy: $F=-\log(Z)$
- entropy $S=-\sum_{G(n)} P(g)\log(P(g))$ (which is maximized by our choice of P)
- $<x>=\frac{\mathrm{d}F}{\mathrm{d}\beta}\big|_{\beta=0}$

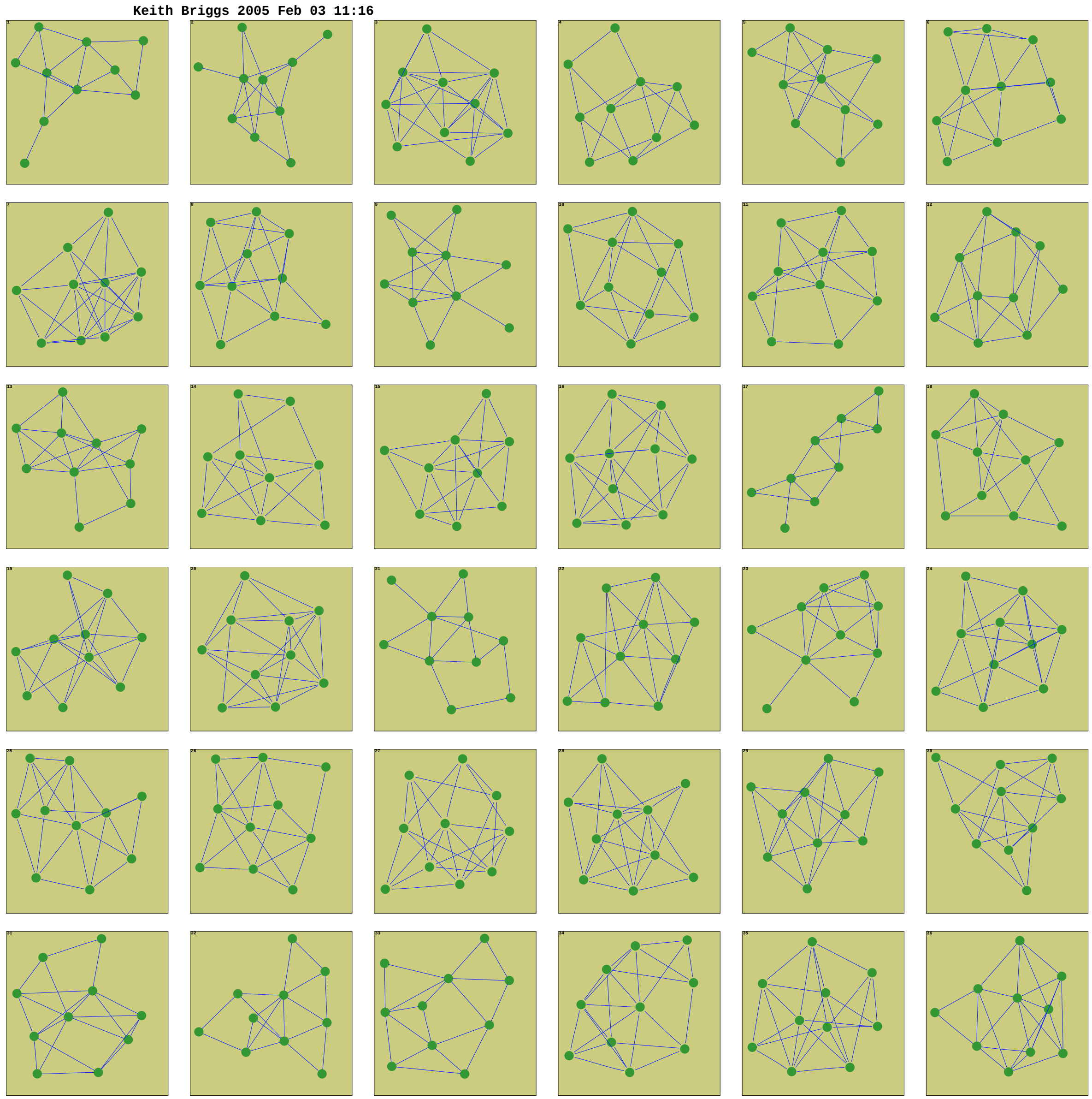
Exactly soluble example - Bernoulli model $G\{n,p\}$

- $G\{n,p\}$ has n nodes and each possible edge appears independently with fixed probability p
- let $x =$ number of edges
- $<x>= m = \binom{n}{2}p$
- $Z = (1+\exp(-\beta))^{\binom{n}{2}}$
- $F = \binom{n}{2}\log(1+\exp(-\beta))$
- which gives $P(g) = p^m(1-p)^{\binom{n}{2}-m}$ as expected
- i.e. each graph appears with equal probability

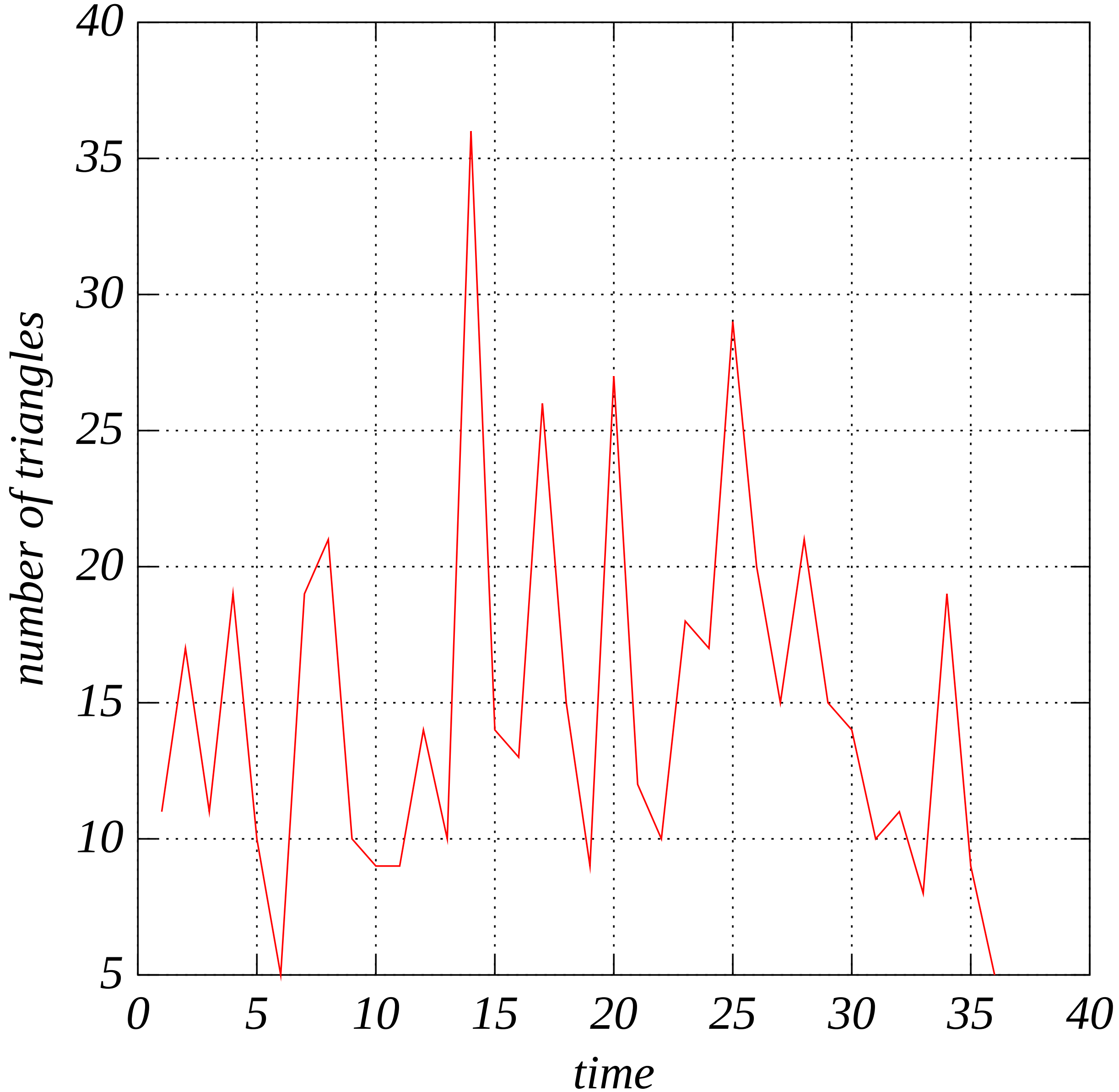
Metropolis simulation

- works by defining a random walk in G which has equilibrium distribution equal to our desired P
- typically we choose a pair of nodes, and then flip its state depending on whether the flip is energetically favourable

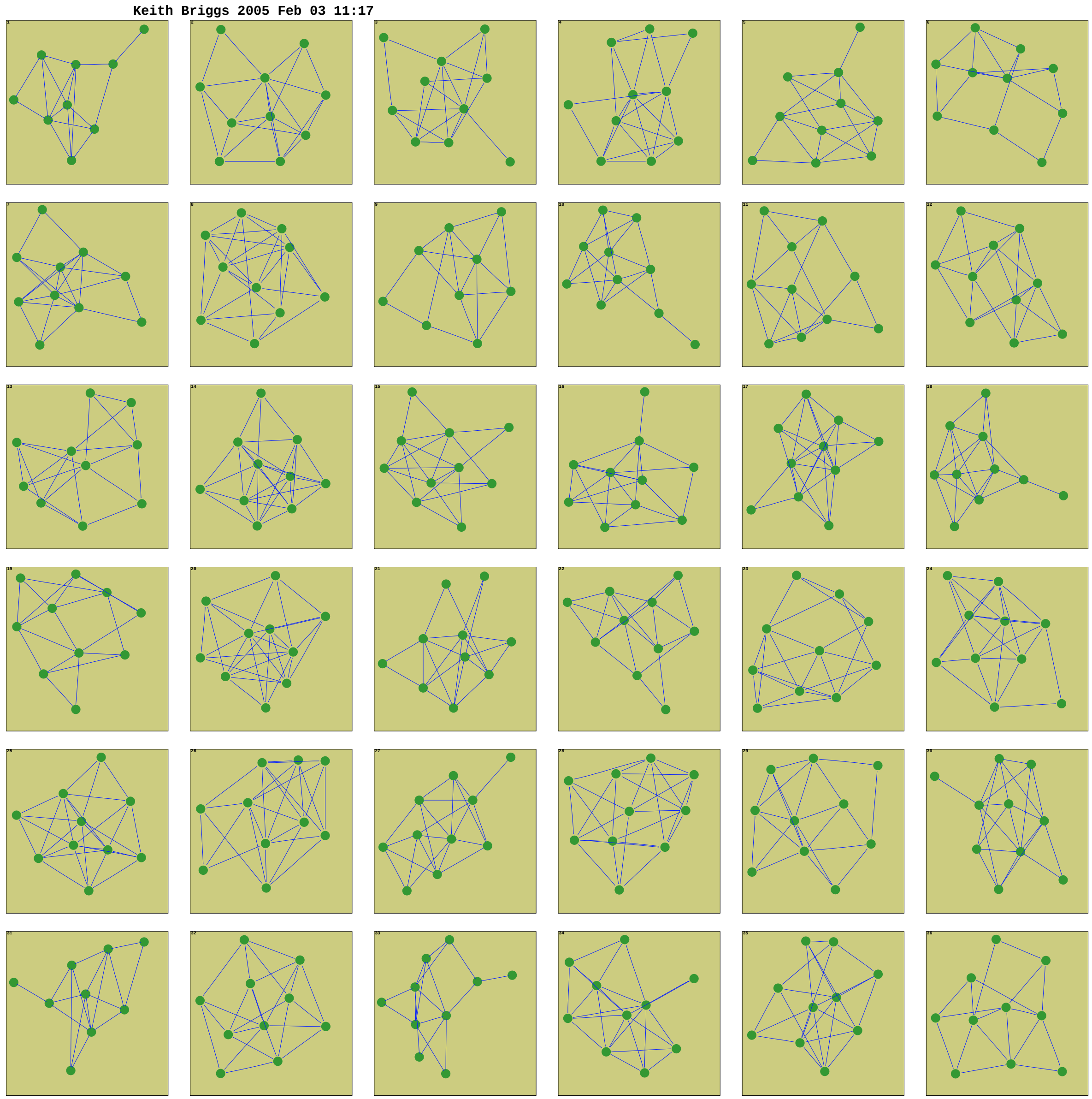
Simulation example 1 - favouring triangles



Simulation example 1 - favouring triangles



Simulation example 2 - favouring degree 3



Some references (amongst many)

J. Berg, M. Lässig *Correlated random networks*
http://xxx.soton.ac.uk/abs/cond-mat/0205589

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