(This appendix is included merely as a historical curiosity. It was the first article I ever wrote on the subject, and the one I still get most requests for! It is [Briggs, 1989])

0.1 Introduction

The family of functions $f_a : \mathbb{R} \to \mathbb{R}$ (dependent on the parameter $a \in \mathbb{R}$) defined by the map

$$x \rightarrowtail f_a(x) = a - x^2 \tag{0.1}$$

has the property that there exist critical values a_i^* of a, at which bifurcations occur in the sets of limit points of sequences $\{x_i\}$ defined by the iteration

$$x_{i+1} = f_a(x_i), \qquad i = 0, 1, 2, \ldots; \qquad x_0 < \sqrt{a}. \tag{0.2}$$

If the set of limit points for a given a has n elements, we describe the iteration as having an n-cycle. In other words, the sequence x_i is asymptotically periodic with period n. There exist cycles of each integer period [Keener, 1986]; amongst these we are especially interested in the superstable n-cycles, that is, those that contain 0 as one of the cycle points. Since the maximum of f occurs at 0, it follows that the stability

$$\Lambda_n(a) \equiv \prod_{i=0}^{n} \frac{df_a(x_i)}{dx} \tag{0.3}$$

is zero at a superstable n-cycle.

Let a_i^* be the least value of a at which a bifurcation to period 2^i occurs. It is known from the work of Feigenbaum [Feigenbaum, 1980*d*] that the sequence

$$\delta_i \equiv \frac{a_{i-1}^* - a_{i-2}^*}{a_i^* - a_{i-1}^*}, \qquad i = 2, 3, 4, \ldots \tag{0.4}$$

is convergent to a value $\delta \approx 4.669$. We describe in this note a direct method of calculation of δ. Previous methods have either used a numerical search for bifurcation values a_i^*, which is unreliable because the limited precision of computer arithmetic introduces artificial periods into the sequence x_i, or methods using power series approximations [Feigenbaum, 1980*d*]. The number δ is of interest in several physical and biological problems [Cvitanović, 1984; Briggs, 1987] which are modelled by equation (0.1). For example, it is equivalent to the logistic equation of population dynamics. Of course in practice a few digits of δ are sufficient. Nevertheless, the problem of calculating δ to many places has the same fascination as did the calculation of π to earlier generations of mathematicians.

0.2 The direct method

We consider the sequence of polynomials in a defined by

$$b_k(a) = a - [b_{k-1}(a)]^2, \qquad k = 1, 2, 3, \ldots \tag{0.5}$$

$$b_0(a) = 0. \tag{0.6}$$

The following property makes these polynomials useful for our purposes.

Lemma 1 *Let $k = 2^n$. Then f_a has a superstable k-cycle iff $b_k(a) = 0$.*

The proof is trivial. Thus superstable 2^n-cycles occur at zeros of b_{2^n}. We denote by a_i the least parameter value at which a superstable 2^i-cycle occurs. Clearly a bifurcation value a_i^* must occur between a_{i-1} and a_i. We conjecture that the stars can be removed in equation (0.4) without change to the limit δ, although we do not attempt to prove this. We will instead calculate as if δ were defined with superstable values a_i in equation (0.4), and see whether our δ agrees with that given by Feigenbaum.

Thus we can calculate δ to arbitrary precision by locating zeroes of the polynomials b_k. For this purpose Newton's method is satisfactory, so that the complete method is:

$$a_i^0 = a_{i-1} + \frac{a_{i-1} - a_{i-2}}{\delta_{i-1}}, \qquad i = 2, 3, 4, \ldots \tag{0.7}$$

$$a_i^{j+1} = a_i^j - \frac{b_{2^i}(a_i^j)}{b'_{2^i}(a_i^j)}, \qquad j = 0, 1, 2, \ldots \tag{0.8}$$

$$b'_k(a) = 1 - 2b'_{k-1}(a)b_{k-1}(a), \qquad k = 1, 2, 3, \ldots \tag{0.9}$$

$$a_i = \lim_{j\to\infty} a_i^j \tag{0.10}$$

$$\delta_i = \frac{a_{i-1} - a_{i-2}}{a_i - a_{i-1}} \tag{0.11}$$

$$\delta = \lim_{i\to\infty} \delta_i \tag{0.12}$$

The first equation produces an initial approximation to the next superstable a value, which is refined by the Newton iteration; b'_k is the derivative of b_k. Thus a_i^j is a sequence convergent to the ith zero of b_{2^i}. The process was programmed in Turbo Pascal version 4.0, using extended precision, and started with $a_0^0 = 0; a_1^0 = 1; b'_0 = 0$ and $\delta_1 = 3.2$.

The rate of convergence of appears to be roughly linear, so that about one more significant decimal digit is gained every two iterations. The results were:

i	a_i	delta_i
2	1.3107026413	3.21851142203809
3	1.3815474844	4.38567759856834
4	1.3969453597	4.60094927653808
5	1.4002530812	4.65513049539198
6	1.4009619629	4.66611194782857
7	1.4011138049	4.66854858144684
8	1.4011463258	4.66906066064834
9	1.4011532908	4.66917155537963
10	1.4011547825	4.66919515602875
11	1.4011551020	4.66920022907521
12	1.4011551704	4.66920131316059
13	1.4011551851	4.66920154839814

The algorithm depends on finding the correct zero by Newton's method of a high degree polynomial with many closely spaced zeros. Thus it will fail if the initial approximation is not close enough to the required zero. This is the limiting factor determining the maximum precision of the above results.

0.3 Feigenbaum's α

If d_k denotes the value of the nearest cycle element to 0 in the superstable k-cycle, then the sequence

$$\alpha_i = \frac{d_i}{d_{i+1}}, \qquad i = 1, 2, 3, \dots \tag{0.13}$$

is convergent to a value about 2.502. This constant is most easily calculated by realizing that the derivative b' defined above must satisfy

$$\lim_{i\to\infty} \frac{b'_{i+1}(a_{i+1})}{b'_i(a_i)} = \delta/\alpha. \tag{0.14}$$

(To see this, consider the slope of the line joining successive 'corners' of the graph of the figure.) Taking the calculation as far as b_{20} gave $\alpha = 2.502907875095$.

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