How to count without counting

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The eprint

- *Loglog counting of large cardinalities*
- Marianne Durand and Philippe Flajolet
- *Engineering and applications track* of the *11th Annual European symposium on algorithms* (ESA 2003, Budapest Sept 15-20)
- to be published by Springer, Lecture Notes in Computer Science
- `algo.inria.fr/flajolet/Publications/DuFl03.ps.gz`

Algorithms

- A precisely defined, provably correct (for valid inputs) computational procedure for a specific problem
- Abu ʿAbd Allâh Muḥammad ibn Mûsâ al-Khwârizmî
  - *born: about 780 in Baghdad*
  - *died: about 850*
- e.g. Euclid's algorithm for the greatest common divisor of two positive integers:

```
def gcd(x,y):
  while y:
    y,x=x%y,y
  return x
```

Types of computational procedure

★ deterministic algorithm

- ▷ *always returns the same output for the same input*
- ▷ *output proved always correct*
- ▷ *always terminates in finite time*
- ▷ *involves no random (stochastic) steps*

★ heuristic

- ▷ *not proved to always return the correct result*
- ▷ *usually involves some 'rules of thumb' - arbitrary but reasonable-looking steps*
- ▷ *not proved to terminate in finite time*
- ▷ *an 'engineering' solution*

★ stochastic algorithm

- ▷ *output proved usually correct, within certain probabilistic bounds*
- ▷ *may involve random (stochastic) steps*
- ▷ *may be much faster than a deterministic algorithm for the same problem*

A deterministic counting algorithm

★ Problem: given a multiset M (a collection of objects, possibly with repeats), determine how many different objects there are in M

★ obvious algorithm:

- set $D = \{\}$ (the empty set)
- for each x object in M...
  - ▷ *see if x is in D, and if not, add it to D*
- count the numbers of elements in D, and return it

★ D is a list which grows, so a lot of time is wasted in memory allocation

★ as D becomes large, it becomes slower and slower to find whether a given x is in D

★ can we do better with a stochastic algorithm?

The Durand and Flajolet algorithm 1

- define $\rho(b_1 b_2 b_3 \dots) \equiv \text{argmin}_k\{k \text{ such that } b_k = 1\}$
- choose parameter k (typically 10 to 16)
- $m = 2^k$, buckets $M_1, M_2, M_3, \dots, M_m$, initialized to 0
- $h =$ a hash function (e.g. 32 bits)
- for each word x in the file:
  - $y = h(x)$
  - *j=value of first k bits of y*
  - *l=value of last (hash size$-k$) bits of y*
  - *set M_j to the maximum of M_j and $\rho(l)$*
- size estimate is $E = m\left[\Gamma(-1/m)\frac{2^{-1/m}-1}{\log 2}\right]^{-m} 2^{(\sum_j M_j)/m-1}$

The Durand and Flajolet algorithm 2

- buckets need to be only about $\log\log(n_{\mathsf{max}})$ bits
- E is unbiased:
  - *as* $n \to \infty$, $<E>/n = 1+\theta_1+o(1)$
  - $|\theta_1| < 10^{-6}$
- the standard error S (the standard deviation divided by n) of E satisfies
  - *as* $n \to \infty$, $S = \beta_m/\sqrt{m}+\theta_2+o(1)$
  - $|\theta_2| < 10^{-6}$
  - $\beta_m \approx 1.3$
- practical formula: $S \approx 1.3/\sqrt{m} = 1.3\times 2^{-k/2}$
- an improved version has $S \approx 1.05\times 2^{-k/2}$

The function ρ is easily implemented in C:

```
/* index of first 1 bit in x, counting from leftmost=0 */
unsigned int rho(int x) {
  for (int i=0; i<32; i++) {
    if (x<0) return i;
    x<<=1;
  }
  return 32;
}
```

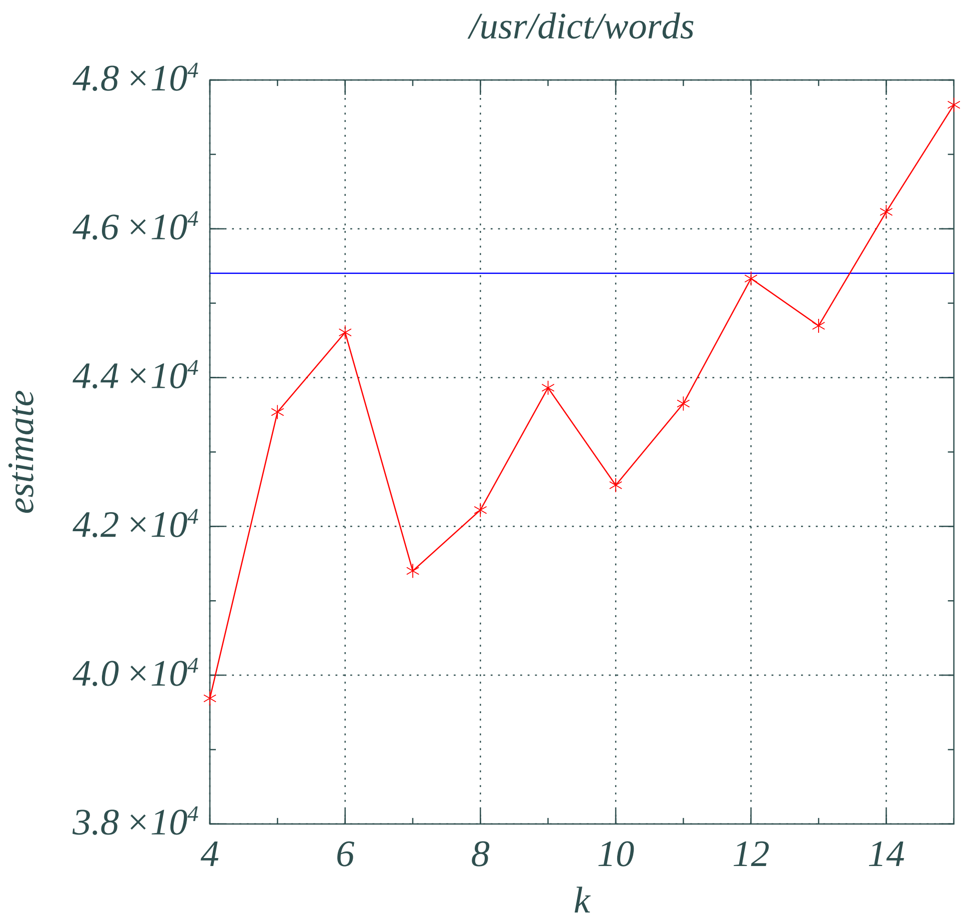
Hash

In this context, a *hash function* is a mapping from $\{0,1\}^n$ to itself with the properties:

- *it is bijective: injective (one-to-one) and surjective (onto)*
- *it has high entropy (on average, close inputs map to distant outputs)*

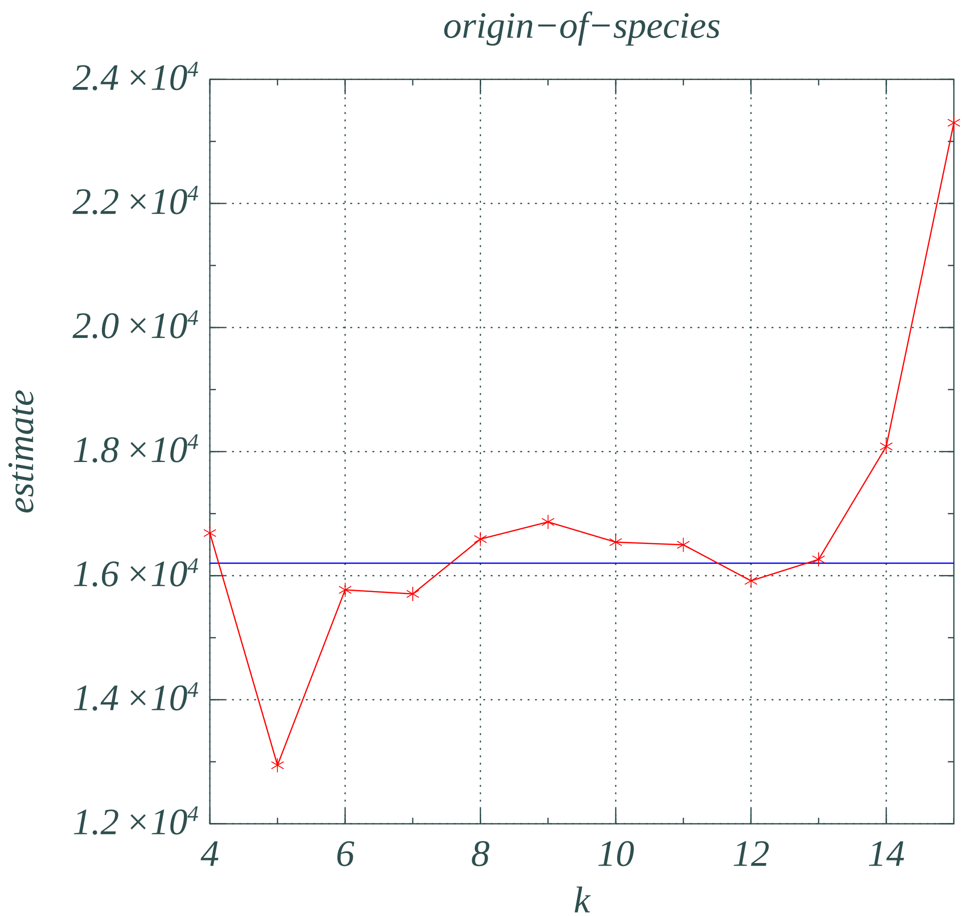
```
unsigned int hash(unsigned int x) {
 x +=~(x << 15);
 x ^=  (x >> 10);
 x +=  (x << 3);
 x ^=  (x >> 6);
 x += ~(x << 11);
 x ^=  (x >> 16);
 return x;
}
```

Results 1: English dictionary



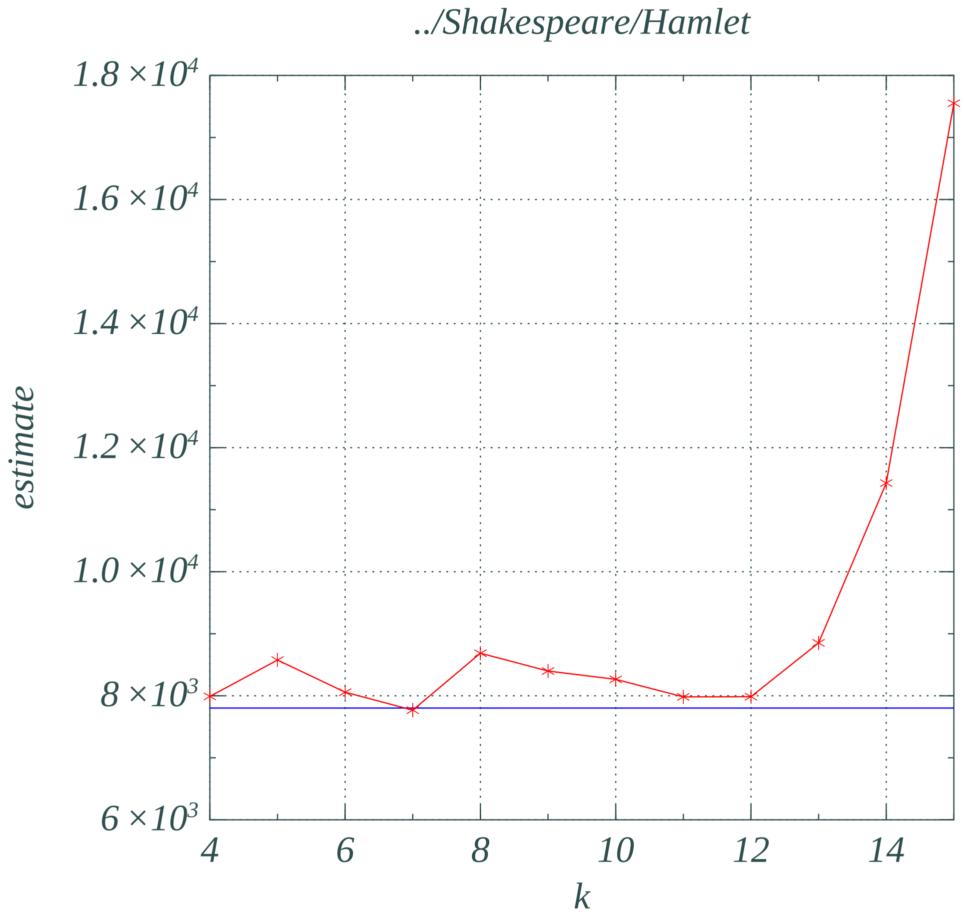
'aardvark aback abaft abandon abandoned abandoning abandonment abandons . . . '

Results 2: Darwin, Origin of species



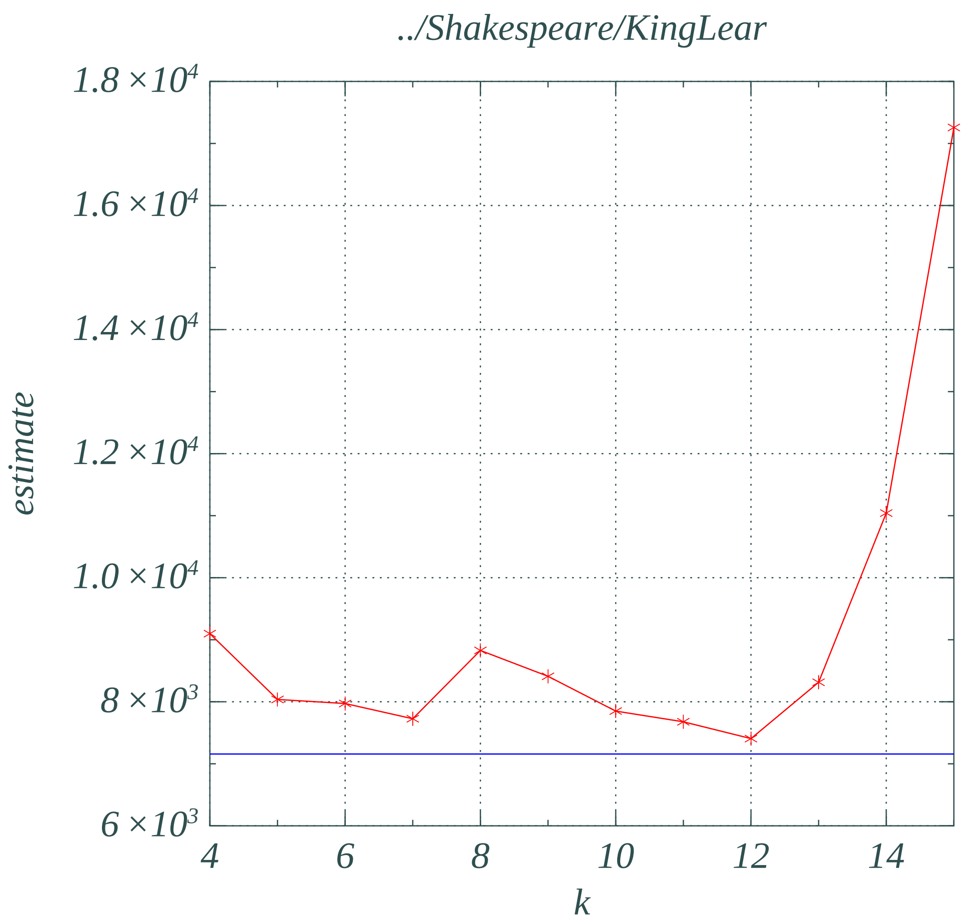
'When on board H.M.S. Beagle as naturalist, I was much struck with certain facts in the distribution of the organic beings inhabiting South America,. . . '

Results 3: Hamlet



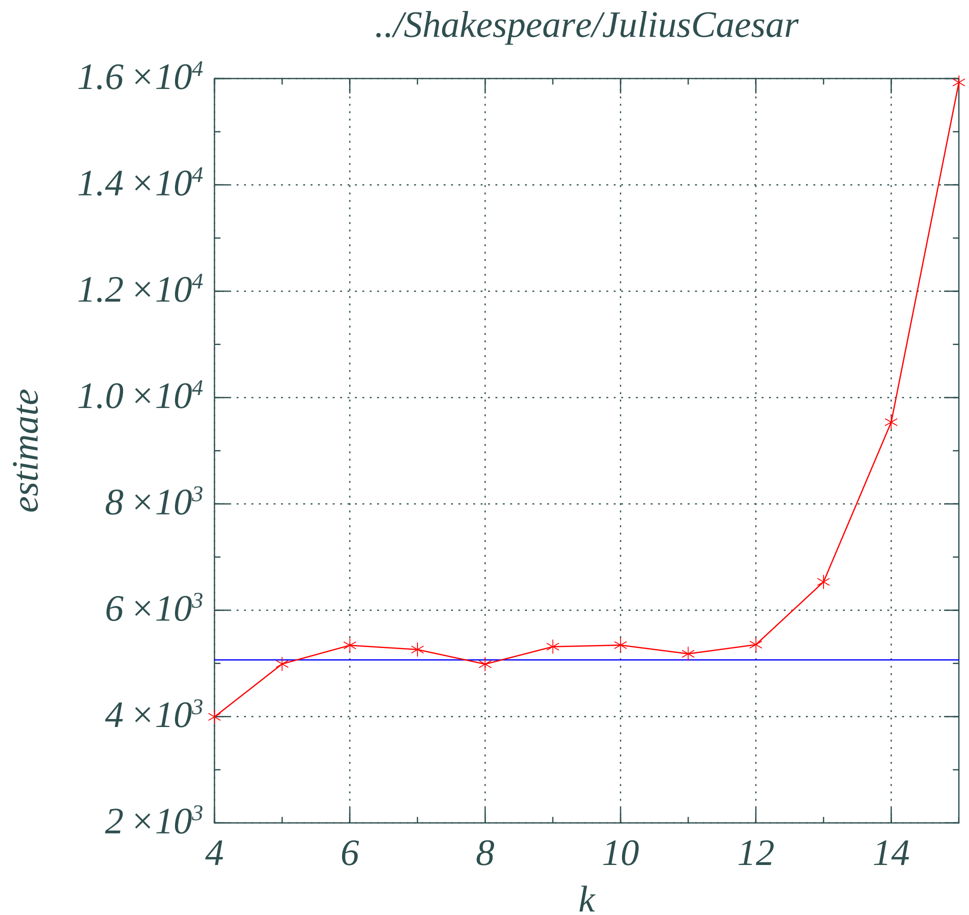
'To be, or not to be: that is the question:
Devoutly to be wish'd. To die, to sleep;. . . '

Results 4: King Lear



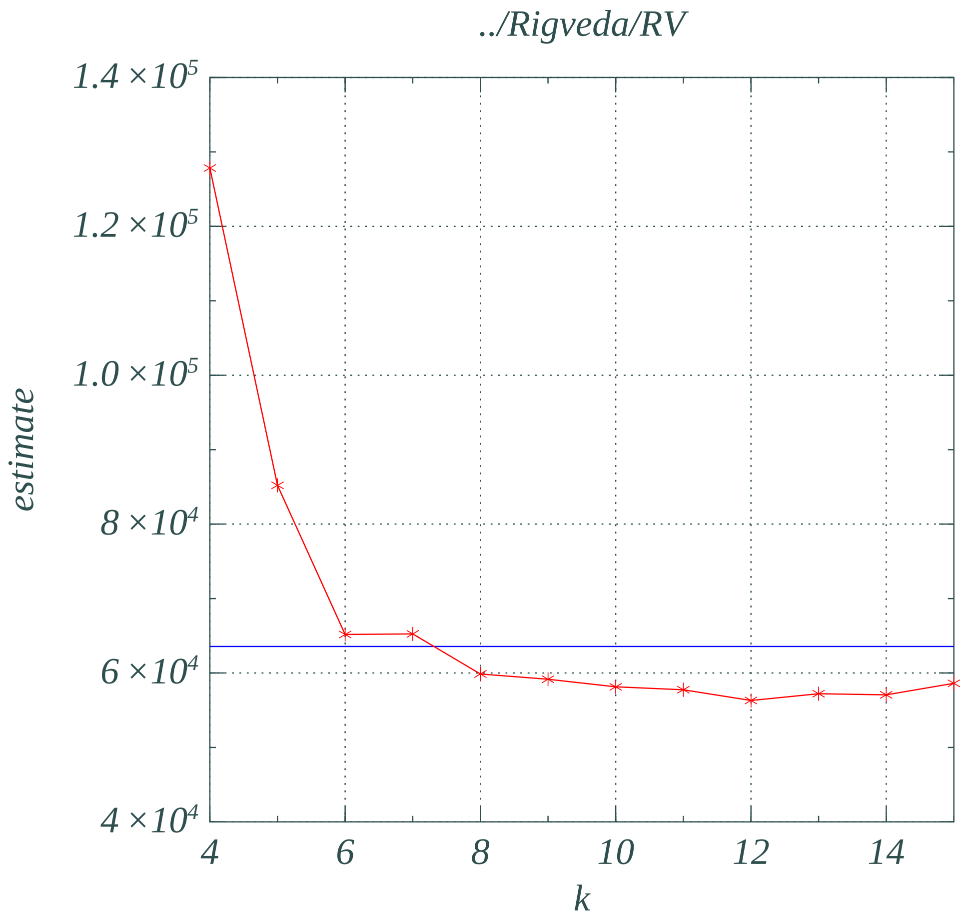
'Kent: I thought the king had more affected the Duke of Albany than Cornwall. . . . '

Results 5: Julius Caesar



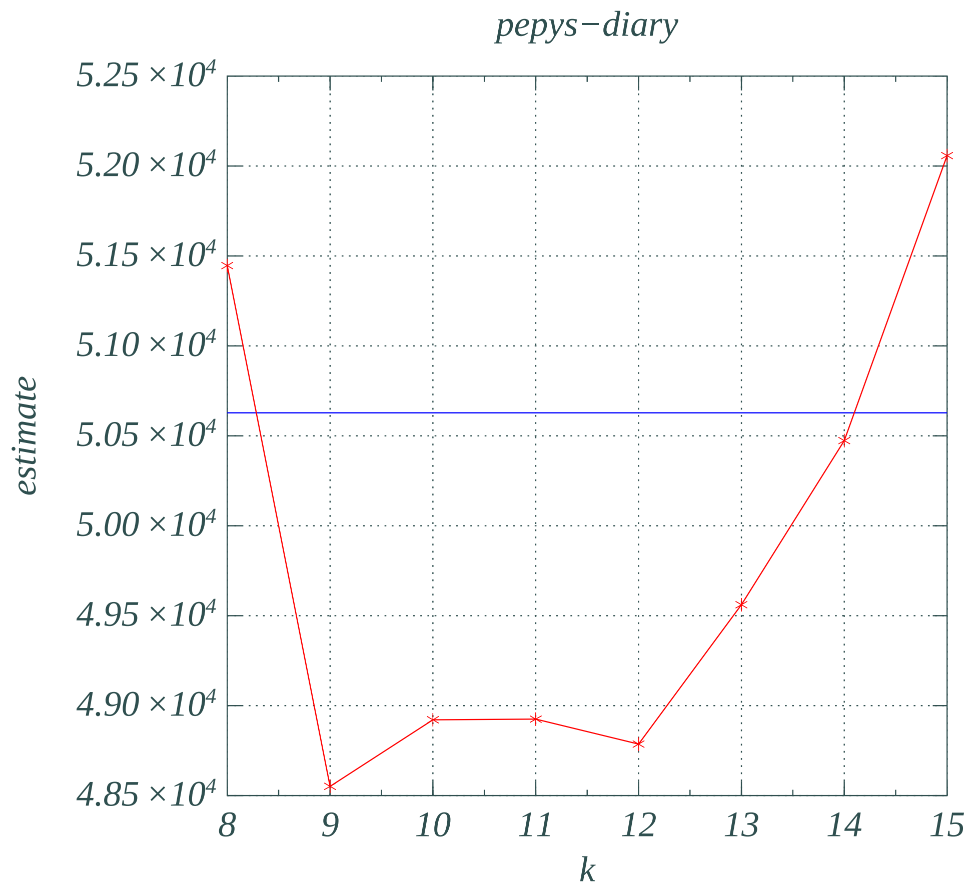
'Scene I. Rome. A street. Enter Flavius, Marullus, and certain Commoners. . . . '

Results 6: Ṛgveda



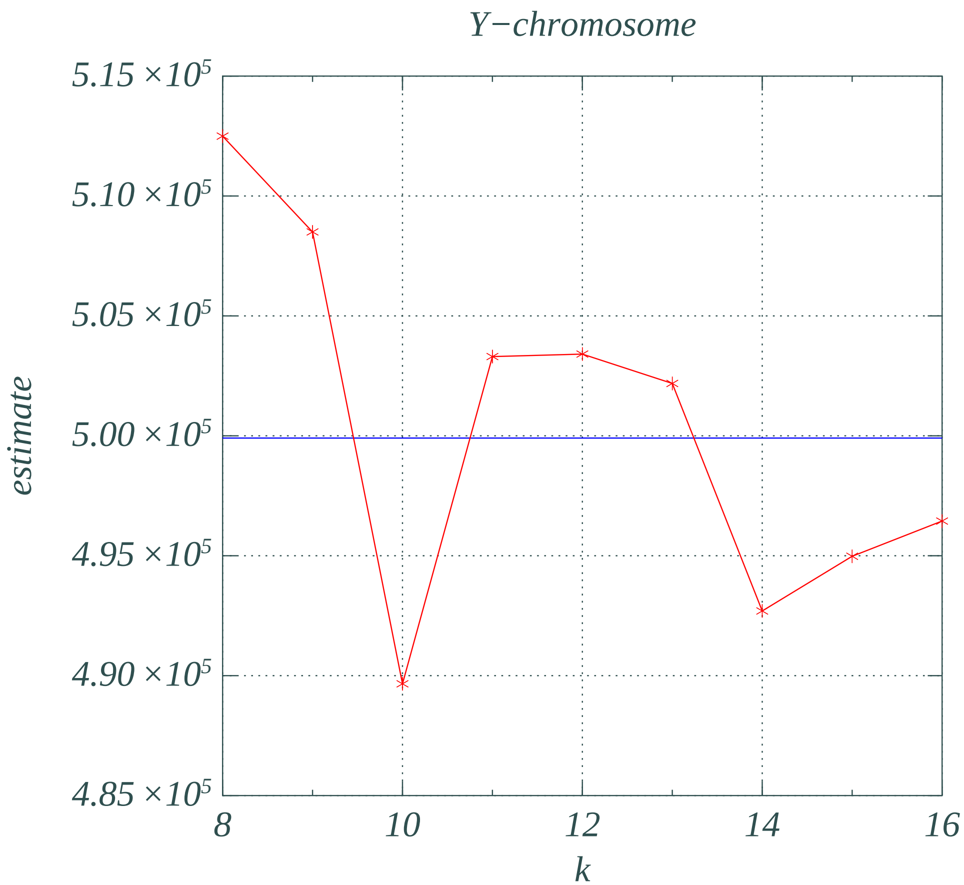
tvám agne dyúbhis tvám āśuśukṣáṇis tvám adbhyás tvám áśmanas pári; tváṃ vánebhyas tvám óṣadhībhyas tváṃ nṛṇā́ṃ nṛpate jāyase śúciḥ; távāgne hotráṃ táva potrám ṛtvíyaṃ táva neṣṭráṃ tvám agníd ṛtāyatáḥ . . .

Results 7: Pepys' diary



'17th. Up, and with my wife, setting her down by her father's in Long Acre, in so ill looked a place, among all the whore houses. . . '

Results 8: Y-chromosome (word=block of 16 codons)



'GAATTCTAGGCTTTCTTTGAAGAGGTAGTAATCTGTAGCCCTCACCTAGG. . . '

Conclusion

If approximate counts are sufficient, they may be obtained very rapidly, and with small, constant memory usage and with known standard error