A precise computation of the Gauss-Kuzmin-Wirsing constant
(preliminary report)

Keith Briggs

Keith.Briggs@bt.com

`http://research.btexact.com/teralab/keithbriggs.html`

2003 July 14 10:52

TYPESET 2003 JULY 14 10:52 IN PDFLATEX ON A LINUX SYSTEM

Introduction

The Gauss-Kuzmin-Wirsing constant determines the asymptotic rate of convergence to the invariant distribution $\log_2(1+x)$ of the Gauss continued fraction map.

It has been previously computed to about 30 decimal places by Flajolet and Vallée and to 100 places by Sebah (both unpublished).

Method

The basic discretization idea was developed by Babenko and Flajolet and Vallée. I work on the interval $[0,1]$ and discretize by computing the action of the Gauss map on Taylor expansions about $1/2$. I then use a QR iteration methid to find the largest 3 or 4 eigenvalues. The largest should be 1 which provides a check. I use the mpfr library for high-precision floating point arithmetic.

The matrix required for truncation order n is

$$M_{jk} = \frac{(-1)^j}{j!(-2)^k} \sum_{i=0}^{k} \binom{k}{i} (-2)^i (i+2)^{\overline{j}} \ \left[\zeta(i+j+2)(2^{i+j+2}-1) - 2^{i+j+2} \right]$$

for $0 \leqslant j, k < n$

Result 1

At $n = 800$ and computing to 1300 bits of precision, I obtained the value

-0.
30366300289873265859744812190155623311087735225365
789518824548146722699529424691098434081193436368
11098272263710616938474614859745801316065265381818
23787913244613989647642974095044629375949048702977
28772511058335175922044472408659119650778105589295
79186714752925653642591844121784234492057255294269
10040657788006767324303643964013896927671340737822
86711534915435462112848419717968

This value has been confirmed by calculation at $n = 1000$ and 1600 bits. It is probably accurate to 385 decimals.

Result 2 - 2003 July 11

At $n = 1200$ and computing to 2000 bits of precision, I obtained the value

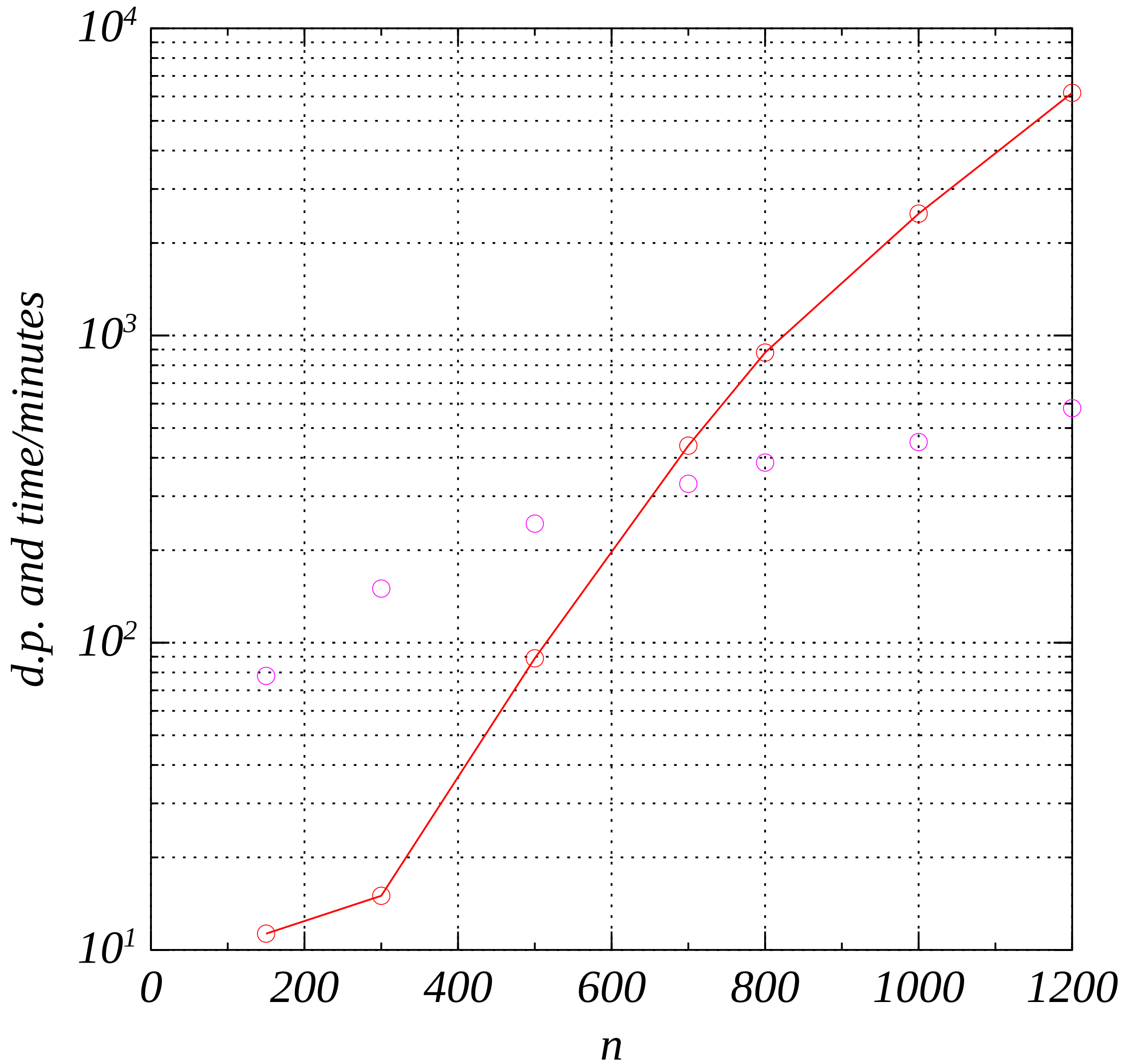
-0.
30366300289873265859744812190155623311087735225365
789518824548146722699529424691098434081193436368
11098272263710616938474614859745801316065265381818
23787913244613989647642974095044629375949048702977
28772511058335175922044472408659119650778105589295
79186714752925653642591844121784234492057255294269
10040657788006767324303643964013896927671340737822
86711534915435462112848419717968616143405897758499
10883150855441132114726744751359199277890841650339
3062695154588788227

This value is probably accurate to 468 decimals.

More precise calculations are in progress.

Timings

Correct decimal places (purple circles) and CPU time in minutes (solid red line) on a 2GHz Pentium vs. truncation order n:



Principal references

E. Weisstein: Gauss-Kuzmin-Wirsing constant `mathworld.wolfram.com/Gauss-Kuzmin-WirsingConstant.html`

S. Finch: Gauss-Kuzmin-Wirsing constant *Mathematical constants*, CUP 2003, pp. 151-6

E. Wirsing, *On the theorem of Gauss-Kuzmin-Lévy and a Frobenius-type theorem for function spaces*, Acta Arithmetica **24** 507-528 (1974); MR 49 2637

K. I. Babenko, *On a problem of Gauss*, Soviet Math. Dokl. **19** 136-140 (1978); MR 57 12436

P. Flajolet and B. Vallée, *Continued Fractions, Comparison Algorithms, and Fine Structure Constants*; in 'Constructive, Experimental, and Nonlinear Analysis', 27, CMS Conf. Proc., M. Théra Ed., (Amer. Math. Soc., Providence), pp. 53-82.

P. Flajolet and B. Vallée, *On the Gauss-Kuzmin-Wirsing constant*, unpublished note (1995) data from Plouffe site

Further references (thanks to Steven Finch for these!)

E. Wirsing, On the theorem of Gauss-Kuzmin-Lévy and a Frobenius-type theorem for function spaces, *Acta Arithmetica* 24 (1974) 507-528; MR 49 #2637

D. E. Knuth, *The Art of Computer Programming*, v. 2, *Seminumerical Algorithms*, 2nd ed., Addison-Wesley, 1981; MR 44 #3531.

K. I. Babenko, On a problem of Gauss, *Soviet Math. Dokl.* 19 (1978) 136-140; MR 57 #12436.

K. I. Babenko and S. P. Iurev, On the discretization of a problem of Gauss, *Soviet Math. Dokl.* 19 (1978) 731-735; MR 81h:65015.

A. Durner, On a theorem of Gauss-Kuzmin-Lévy, *Arch. Math.* 58 (1992) 251-256; MR 93c:11056.

A. J. MacLeod, High-accuracy numerical values in the Gauss-Kuzmin continued fraction problem, *Computers Math. Applic.* v. 26, n. 3 (1993) 37-44; MR 94h:11114.

H. Daudé, P. Flajolet and B. Vallée, An average-case analysis of the Gaussian algorithm for lattice reduction, *Combin. Probab. Comput.* 6 (1997) 397-433; INRIA preprint RR2798; MR 99a:65196.

P. Flajolet and B. Vallée, On the Gauss-Kuzmin-Wirsing constant, unpublished note (1995).

P. Flajolet and B. Vallée, Continued fraction algorithms, functional operators, and structure constants, *Theoret. Computer Sci.* 194 (1998) 1-34; INRIA preprint RR2931; MR 98j:11061.

J. C. Lagarias, Number theory and dynamical systems, *The Unreasonable Effectiveness of Number Theory*, Proc. 1991 Orono conf., ed. S. A. Burr, Amer. Math. Soc., 1992, pp. 35-72; MR 93m:11143.

C. Faivre, The rate of convergence of approximations of a continued fraction, *J. Number Theory* 68 (1998) 21-28; MR 98m:11083.

D. Hensley, The number of steps in the Euclidean algorithm, *J. Number Theory* 49 (1994) 142-182; MR 96b:11131.

M. Iosifescu and S. Grigorescu, *Dependence with Complete Connections and its Applications*, Cambridge Univ. Press 1990; MR 91j:60098.

M. Iosifescu, A survey of the metric theory of continued fractions, fifty years after Doeblin's 1940 paper, *Prob. Theory Math. Stat.* 1 (1990) 550-572; MR 93c:11059.

M. Iosifescu, A very simple proof of a generalization of the Gauss-Kuzmin-Lévy theorem on continued fractions, and related questions, *Rev. Roumaine Math. Pures Appl.* 37 (1992) 901-914; MR 94j:40003.

M. Iosifescu, On the Gauss-Kuzmin-Lévy theorem, I, *Rev. Roumaine Math. Pures Appl.* 39 (1994) 97-118; MR 95j:40002.

M. Iosifescu, On the Gauss-Kuzmin-Lévy theorem, II, *Rev. Roumaine Math. Pures Appl.* 40 (1995) 91-105; MR 98a:40002.

M. Iosifescu, On the Gauss-Kuzmin-Lévy theorem, III, *Rev. Roumaine Math. Pures Appl.* 42 (1997) 71-88.

M. Iosifescu, Doeblin and the metric theory of continued fractions: a functional theoretic solution to Gauss' 1812 problem, *Doeblin and Modern Probability*, Proc. 1991 Doeblin conf., *Contemporary Math.* v. 149, Amer. Math. Soc., 1993, pp. 97-110; MR 94h:11073.

M. Iosifescu, A basic tool in mathematical chaos theory: Doeblin and Fortet's ergodic theorem and Ionescu Tulcea and Marinescu's generalization, *Doeblin and Modern Probability*, Proc. 1991 Doeblin conf., *Contemporary Math.* v. 149, Amer. Math. Soc., 1993, pp. 111-124; MR 94h:47015.

M. Iosifescu, The nearest integer continued fraction expansion: an approach in the spirit of Doeblin, *Doeblin and Modern Probability*, Proc. 1991 Doeblin conf., *Contemporary Math.* v. 149, Amer. Math. Soc., 1993, pp. 125-137; MR 94h:11072.

M. Iosifescu, On some series involving sums of incomplete quotients of continued fractions, *Stud. Cerc. Mat.* 48 (1996) 31-36; corrigendum 48 (1996) 146; MR 2000f:40002 and MR 2000f:40003.

S. Lange and H. Trotter, Continued fractions for some algebraic numbers, *J. Reine Angew. Math.* 255 (1972) 112-134; addendum 267 (1974) 219-220; MR 46 #5258 and MR 50 #2086.

E. Bombieri and A. J. van der Poorten, Continued fractions of algebraic numbers, in *Computational Algebra and Number Theory*, ed. W. Bosma and A. J. van der Poorten, Kluwer, 1995, pp. 137-152; MR 96g:11079.

R. T. Bumby, Hausdorff dimension of sets arising in number theory, *Number Theory: New York 1983-84*, ed. D. V. Chudnovsky, G. V. Chudnovsky, H. Cohn and M. B. Nathanson, Lecture Notes in Math. 1135, Springer-Verlag, pp. 1-8, 1985; MR 87a:11074.

D. Hensley, Metric Diophantine approximation and probability, *New York J. Math*. 4 (1998), 249-257; MR 2000d:11102.

M. Iosifescu, On a 1936 paper of Arnaud Denjoy on the metrical theory of the continued fraction expansion, *Rev. Roumaine Math. Pures Appl.* 44 (1999) 1-17.

K. Dajani and C. Kraaikamp, A Gauss-Kuzmin theorem for optimal continued fractions, *Trans. Amer. Math. Soc.* 351 (1999) 2055-2079; MR 99h:11089.

E. Cesaratto, On the Hausdorff dimension of certain sets arising in number theory, math.NT/9908043.

I. J. Good, The fractional dimensional theory of continued fractions, *Proc. Cambridge Philos. Soc.* 37 (1941) 199-228; corrigenda 105 (1989) 607; MR 3,75b and MR 90i:28013.

D. Hensley, The Hausdorff dimensions of some continued fraction Cantor sets, *J. Number Theory* 33 (1989) 182-198; MR 91c:11043.

D. Hensley, Continued fraction Cantor sets, Hausdorff dimension, and functional analysis, *J. Number Theory* 40 (1992) 336-358; MR 93c:11058.

D. Mayer and G. Roepstorff, On the relaxation time of Gauss' continued-fraction map, I. The Hilbert space approach (Koopmanism), *J. Stat. Phys.* 47 (1987) 149-171; MR 89a:28017.

D. Mayer and G. Roepstorff, On the relaxation time of Gauss' continued-fraction map, II. The Banach space approach (Transfer operator method), *J. Stat. Phys.* 50 (1988) 331-344; MR 89g:58171.

D. H. Mayer, On the thermodynamic formalism for the Gauss map, *Commun. Math. Phys.* 130 (1990) 311-333; MR 91g:58216.

D. H. Mayer, Continued fractions and related transformations, *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, ed. T. Bedford, M. Keane and C. Series, Oxford Univ. Press, 1991; pp. 175-222; MR 93e:58002.

R. Kuzmin, Sur un problème de Gauss, *Atti del Congresso Internazionale dei Matematici*, Bologna, 1928, v. 6, pp. 83-89.

P. Lévy, Sur les lois probabilité dont dépendent les quotients complets et incomplets d'une fraction continue, *Bull. Soc. Math. France* 57 (1929) 178-194.

P. Szüsz, Über einen Kusminschen Satz, *Acta Math. Acad. Sci. Hungar.* 12 (1961) 447-453; MR 27 #124.

D. E. Knuth, *The Art of Computer Programming*, v. 2, *Seminumerical Algorithms*, 3rd ed., Addison-Wesley, 1998, p. 366; MR 44 #3531.

X. Gourdon and P. Sebah, Records of computation (Numbers, Constants and Computation).

P. Flajolet, B. Vallée and I. Vardi, Continued fractions from Euclid to the present day, (2000); preprint.