Notes on the Riemann hypothesis and abundant numbers

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`RH_abundant.tex` ('NOTES' OPTION) TYPESET IN PDFLaTeX ON A LINUX SYSTEM

Introduction

★ In [1], Lagarias showed the equivalence of the Riemann hypothesis (RH) to a condition on harmonic sums, namely

$$\text{RH} \Leftrightarrow \sigma(n) \leqslant H_n + \exp(H_n)\log(H_n)\ \forall n$$

★ Robin [3] had already shown that

$$\text{RH} \Leftrightarrow \sigma(n)/n < e^{\gamma} \log\log(n) \quad \text{for} \quad n > 5040$$

which is probably more convenient for numerical tests

★ so to disprove RH, we need to find n with large $\sigma(n)/n$

Notation

- ⋆ n is always a positive integer; p is always a prime
- ⋆ p_i = ith prime $(p_1\!=\!2, p_2\!=\!3, \ldots)$
- ⋆ harmonic sum for $n \geqslant 1$ is $H_n = \sum_{i=1}^{n} i^{-1}$
- ⋆ sum of divisors is $\sigma(n) = \prod_i \frac{p_i^{a_i+1}-1}{p_i-1}$ for $n = \prod_i p_i^{a_i}$, $(a_i > 0)$
- ⋆ $\sigma(n)/n = \prod_i \frac{p_i - p_i^{-a_i}}{p_i-1}$
- ⋆ I define $\rho(n) \equiv \sigma(n)/n$
- ⋆ $e^\gamma = 1.7810724179901979852365041031071795 4\ldots$, where γ is Euler's constant
- ⋆ we will deal with n too large to represent in the computer $(\approx 10^{(10^{10})})$, but luckily we have its prime factorization which suffices

Superabundant numbers [2]

- ★ n is superabundant (SA) if $\sigma(k)/k < \sigma(n)/n$ for all $k < n$
- ★ if $n = 2^{a_2} 3^{a_3} \dots m^{a_m}$ is SA, where m is the maximal prime factor, then $a_2 \geqslant a_3 \geqslant \cdots \geqslant a_m$
- ★ if $1 < j < i \leqslant m$, then $|a_i - \lfloor a_j \log_i j \rfloor| \leqslant 1$
- ★ $a_m = 1$ unless $n = 4$ or 36
- ★ $i^{a_i} < 2^{a_2+1}$, $i \geqslant 2$
- ★ $m \sim \log(n)$
- ★ SA numbers, with CA numbers in red:
  $2, 2^2, 2.3, 2^2.3, 2^3.3, 2^2.3^2, 2^4.3, 2^2.3.5 \dots$

The work of Robin [3]

- ★ Theorem: **RH** $\Leftrightarrow \rho(n) < e^{\gamma} \log\log(n)$ **for** $n > 5040$
- ★ Theorem: independently of RH, except for $n = 1, 2, 12$:

$$\rho(n) < e^{\gamma} \log\log(n) + \frac{[7/3 - e^{\gamma} \log\log(12)] \log\log(12)}{\log\log(n)}$$

The numerator in the last term is about 0.6482

- ★ But note $\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log(n)} = e^{\gamma}$

The structure of the set of CA numbers

- Defn: n is colossally abundant (CA) if there exists $\epsilon > 0$ such that

$$\sigma(n)/n^{1+\epsilon} \geqslant \sigma(k)/k^{1+\epsilon} \quad \forall k > 1$$

- Robin [3] has given the following precise description, due originally to Erdős & Nicolas [6]
- we first form the set E of *critical* ϵ *values*:

$$E_p \equiv \bigcup_{\alpha=1,2,3,\dots} \left\{ \log_p \left(1 + \frac{1}{\sum_{i=1}^{\alpha} p^i} \right) \right\}$$

$$E \equiv \bigcup_p E_p$$

- we label the elements of E in decreasing order: $\epsilon_1 = \log_2(3/2) > \epsilon_2 = \log_3(4/3) > \epsilon_3 = \log_2(7/6) > \dots$

The structure of the set of CA numbers cotd.

(a) if $\epsilon \notin E$, $\sigma(n)/n^{1+\epsilon}$ has a unique maximum attained at the number n_ϵ with prime exponents given by

$$a_p(\epsilon) = \left\lfloor \log_p \left(\frac{p^{1+\epsilon} - 1}{p^\epsilon - 1} \right) \right\rfloor - 1$$

(b) if ϵ satisfies $\epsilon_{i+1} < \epsilon < \epsilon_i$ for $i = 1, 2, 3, \dots,$ then n_ϵ is constant and we call it N_i. We have $N_1 = 2, N_2 = 6, \dots$

(c) if the sets E_p are pairwise disjoint (which is likely, but not certainly known), then the set of CA numbers is equal to the set of $N_i, i = 1, 2, 3, \dots$. If this is the case, $\sigma(n)/n^{1+\epsilon}$ attains its maximum at the two points N_i and N_{i+1}

(d) if the sets E_p are not pairwise disjoint, then for each $\epsilon_i \in E_q \cap E_p$, $\sigma(n)/n^{1+\epsilon_i}$ attains its maximum at the four points N_i, qN_i, rN_i and $N_{i+1} = qrN_i$

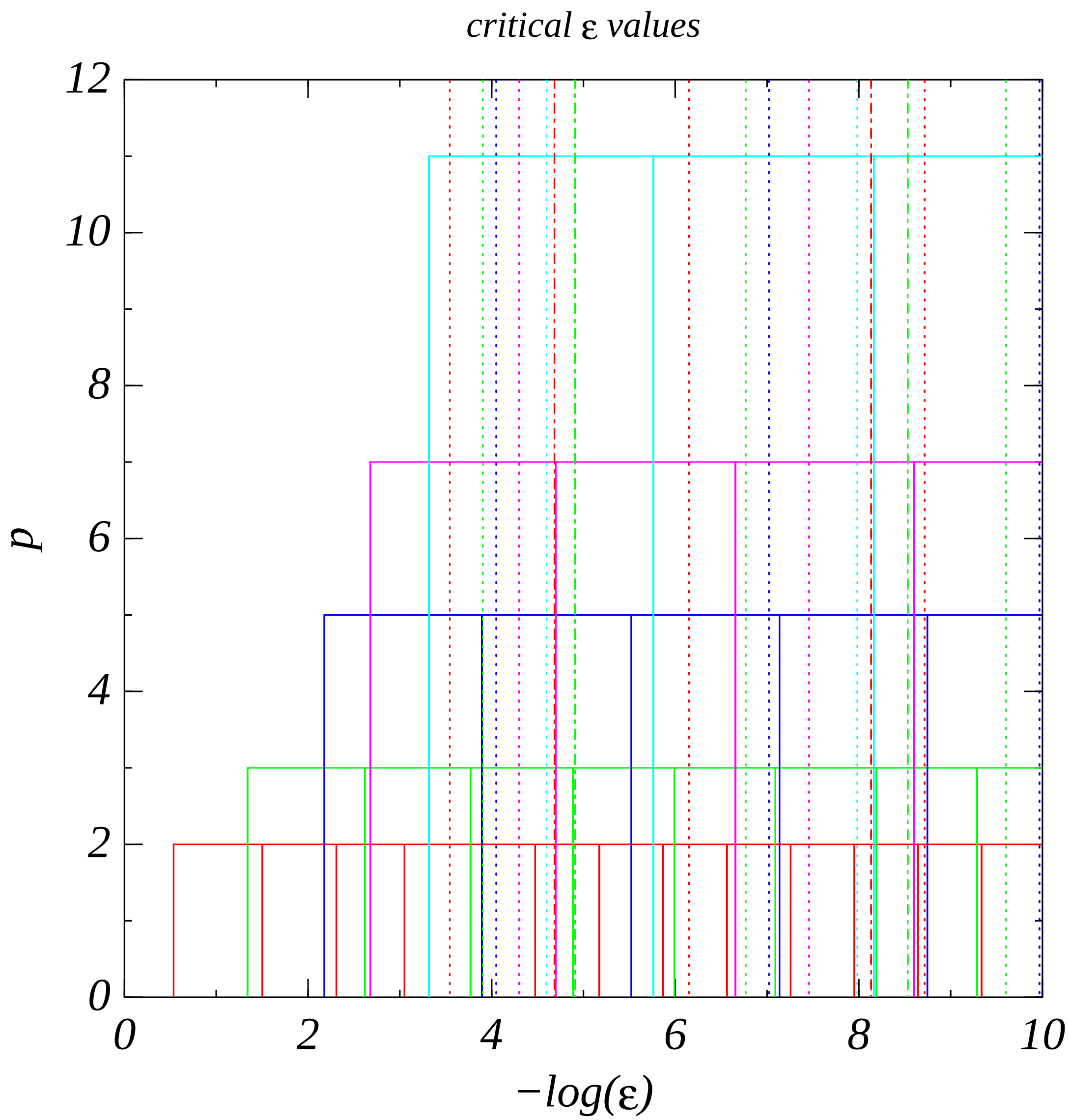
Alaoglu & Erdős [2]

- these authors have a slightly stronger definition:
- Defn: n is colossally abundant if

$$\begin{aligned} \sigma(n)/n^{1+\epsilon} &> \sigma(k)/k^{1+\epsilon} \quad \text{for} \quad 1 \leqslant k < n \\ \sigma(n)/n^{1+\epsilon} &\geqslant \sigma(k)/k^{1+\epsilon} \quad \text{for} \quad k > n \end{aligned}$$

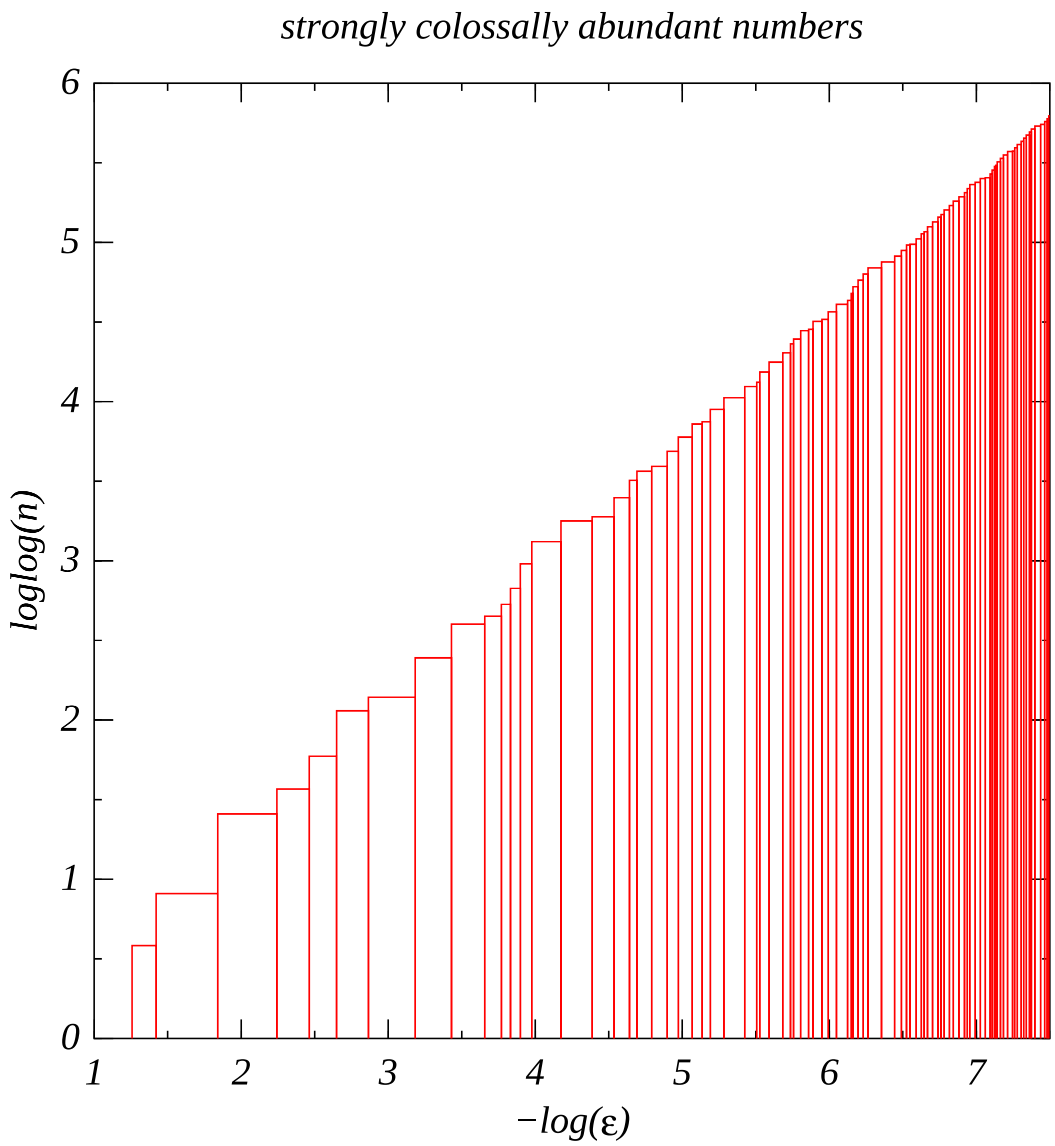
- the effect of this is to make a unique choice from the 2 or 4 possibilities in cases (c) and (d) above. But I will perform my computations with the Robin definition
- we will call these numbers strongly colossally abundant (SCA) if it is necessary to distinguish them from ordinary CA numbers

Position of critical ϵ values



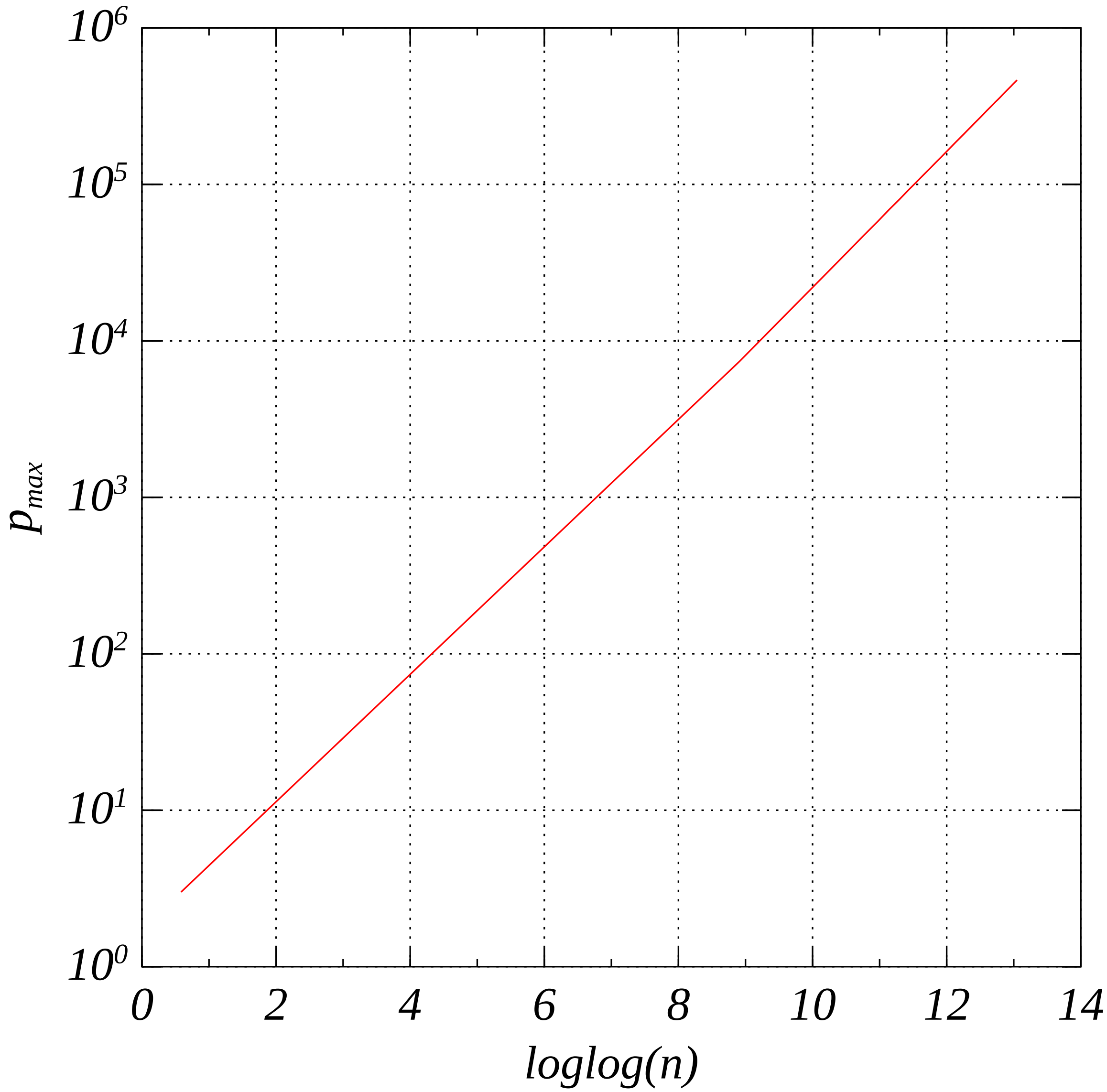
The vertical lines mark the critical ϵ values arising from the small primes on the y axis

Density of critical ϵ values



The vertical lines mark the critical ϵ values

Dependence of maximal prime on n



This shows the maximal prime needed as a function of $\log\log(n)$

Method

- it is known that if RH is false, there will be a violation of Robin's inequality which is a CA number . . .
- for a range of small $\epsilon > 0$, I compute n by the A&E formula
- I compute RHS-LHS of Robin's inequality (let's call this the *deviation* $\delta(n) \equiv e^\gamma \log\log(n) - \rho(n)$), and look for any violations (i.e. $\delta < 0$)
- we can also plot $\eta(n) \equiv e^\gamma - \rho(n)/\log\log(n)$
- the following plots show the behaviour observed so far (to about $\epsilon = \exp(-25)$)

Computational difficulties

- ⋆ the exponents $a_p(\epsilon)$ must be computed in interval arithmetic, to ensure they are correct and not corrupted by roundoff error. This means not just high precision arithmetic, but dynamically varying precision
- ⋆ millions of primes are needed. Typically the n we deal with have a huge tail of many primes to the power 1. It is fastest to precompute primes with a sieve, but then much storage is required.
- ⋆ how do we vary ϵ to not miss any SCA numbers? (DONE)
- ⋆ how to we compute explicit examples of WCA numbers?
- ⋆ there are many other difficulties . . .

Computational strategy

We keep a list z of records, containing: a prime p, $\log p$, its exponent a, and a critical ϵ_{c}, which is the value of ϵ at which this exponent will next change (as ϵ is decreased). We exclude 1 exponents, which are counted by $ones$. We first initialize:

- *fix $0 < \epsilon_{\mathit{start}} \leqslant 1$. Then, for each prime p, compute $a = \left\lfloor \log_p \left(\frac{p^{1+\epsilon_{\mathit{start}}}-1}{p^{\epsilon_{\mathit{start}}}-1} \right) \right\rfloor - 1$ and store it in the z list if $a \geqslant 2$. If $a = 1$, just increment the variable $ones$. Stop when $a = 0$. During this p loop, also update $\log(n)$ and $\rho(n)$*

Then each step of the main loop consists of determining which of possible events A, B, or C occurs:

- *A: a new prime (with exponent 1) is added, so we increment 'ones'. This happens when $\epsilon_{\mathit{ext}} = \log_p(1+p)$ is maximal, where p is the new prime*

- *B: the first prime with exponent 1 has its exponent raised to 2. This happens when $\epsilon_{\mathit{inc}} = \log_p((p+1+1/p)/(p+1))$ is maximal, where p is the prime in question*

- *C: a prime with exponent $\geqslant 2$ has its exponent incremented. This happens when $\epsilon_{\mathit{max}} = \log_p((1-p^{a+1})/(p-p^{a+1}))$ is maximal, where p is the prime in question and a its exponent*

Prime generation

For simple tests, just use a lookup list. With the BERNSTEIN option, `crit_eps14.c` uses Bernstein's quadratic sieve. Here we cannot just look up any prime; rather we have a function to get the next prime and advance the internal state (also to peek at the current prime without advancing). So the strategy is to use 2 prime generators - one (pg1) for the z list (which only grows slowly), and another (pg2) to see if the list of 1s needs extending. pg2 goes a long way, but we rely on Bernstein's code to keep it efficient.

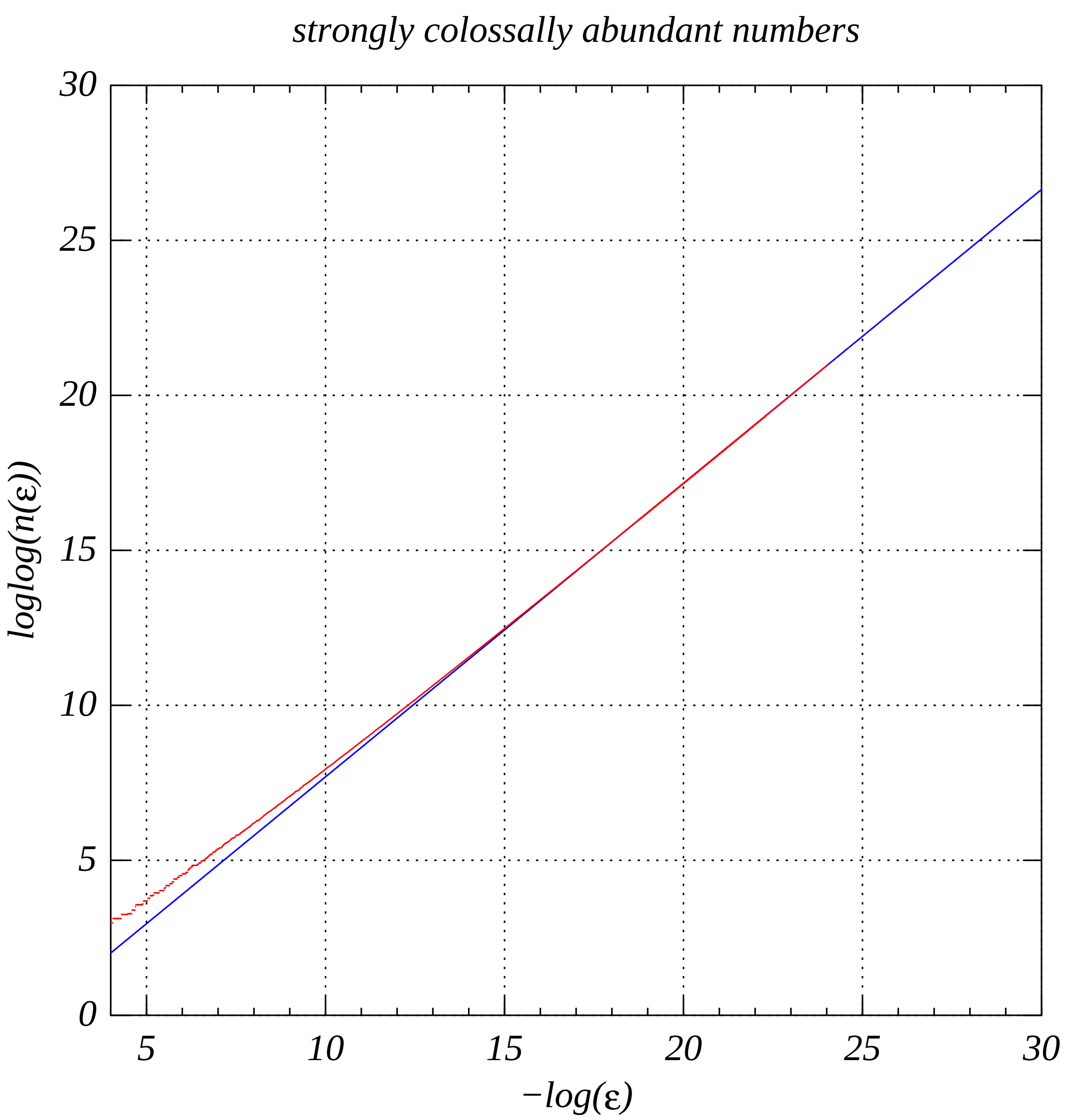
Questions

- ★ how does n depend on ϵ?
- ★ how does the largest prime needed depend on ϵ?
- ★ how does δ depend on ϵ?
- ★ interesting observation: when p is large and ϵ is small (the situation we are interested in), in the exponent formula of Alaoglu and Erdős it is already sufficient to use the first term of a Taylor expansion in ϵ, namely as $\epsilon \to 0^+$,

$$\log_p\left(\frac{p^{1+\epsilon}-1}{p^\epsilon-1}\right) = \log_p\left(\frac{p-1}{\log p}\right) - \log_p(\epsilon) + \mathcal{O}(\epsilon)$$

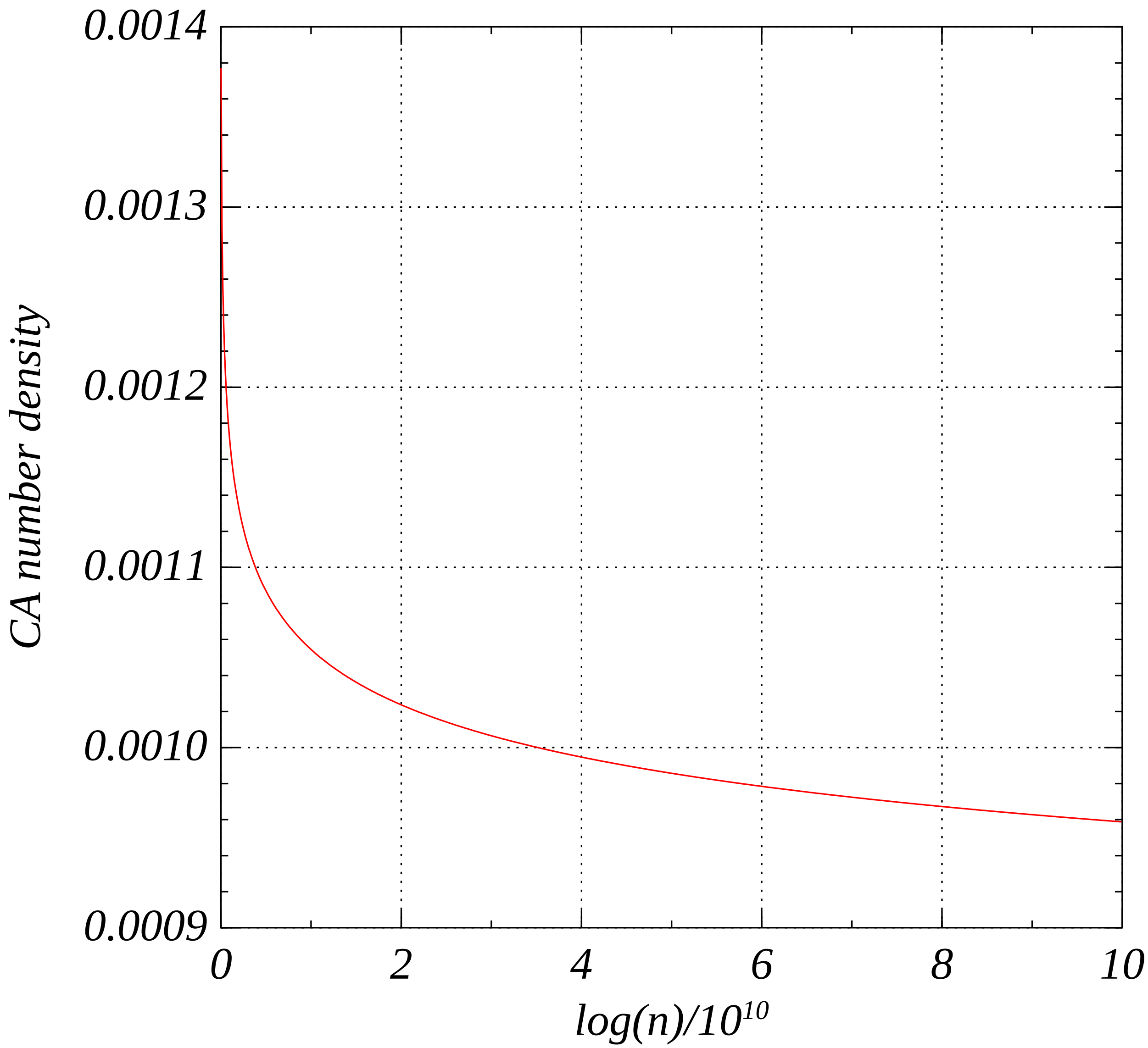
  already has error less than 1/2, so the floor is the correct integer. How can we exploit this?
- ★ in the following plots the computed data is in red and hypothesized fits or trends are in blue

Dependence of n on ϵ

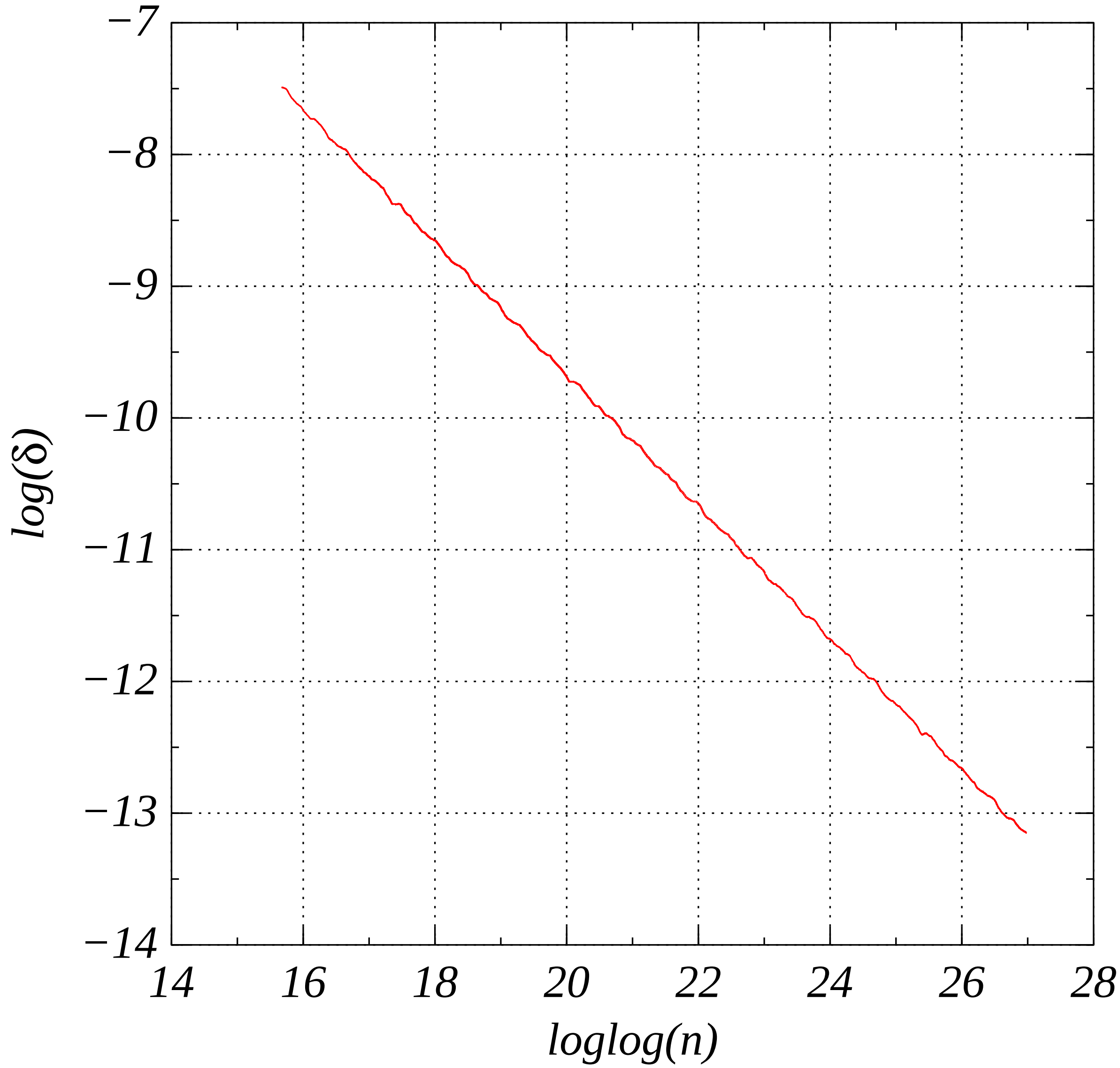


This shows that $\log\log(n)$ at SCA numbers n appears to be asymptotically a linear function of $-\log\epsilon$. The line $-1.779+0.947x$ is guesswork

Density of CA numbers

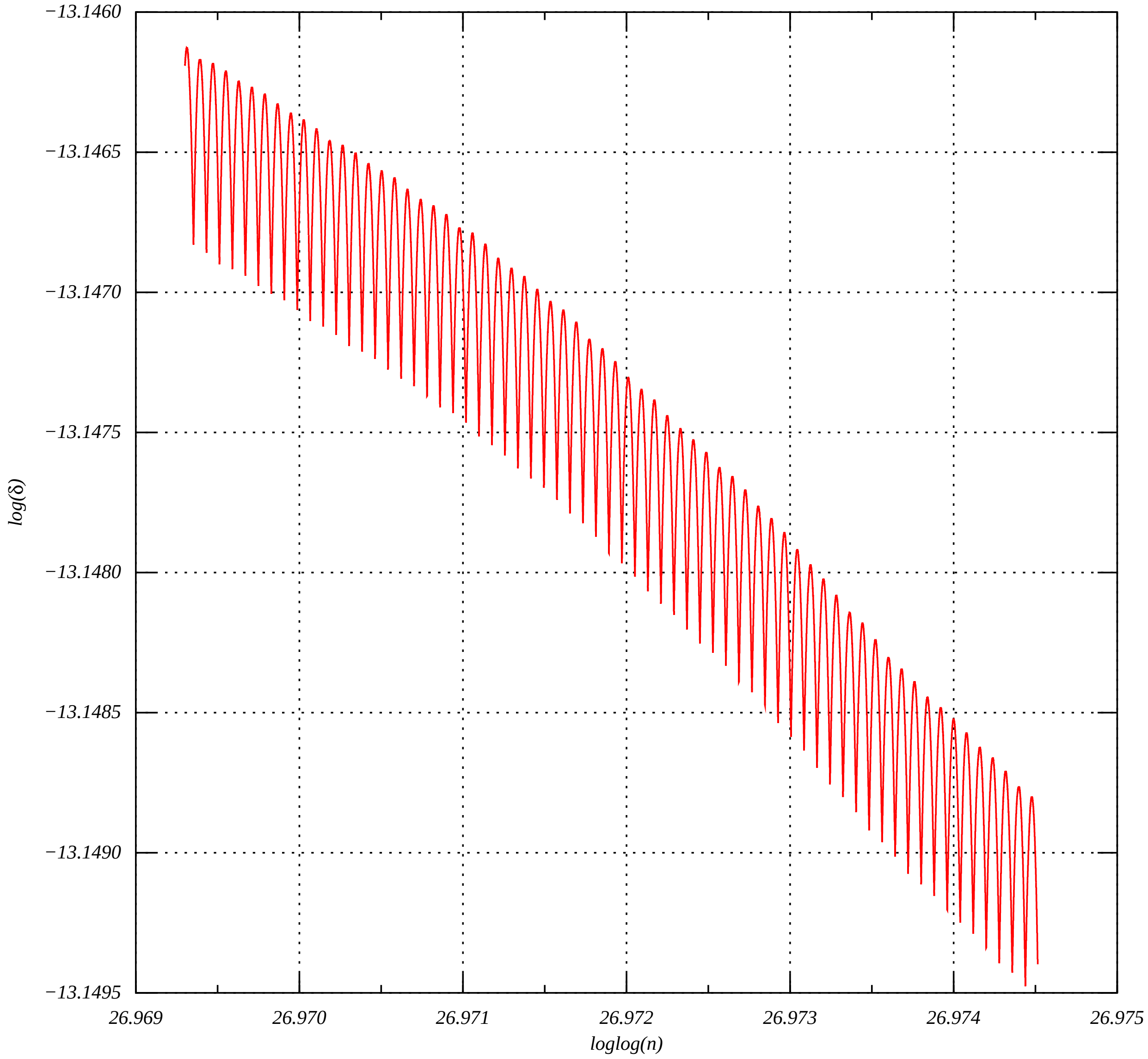


Dependence of δ on n



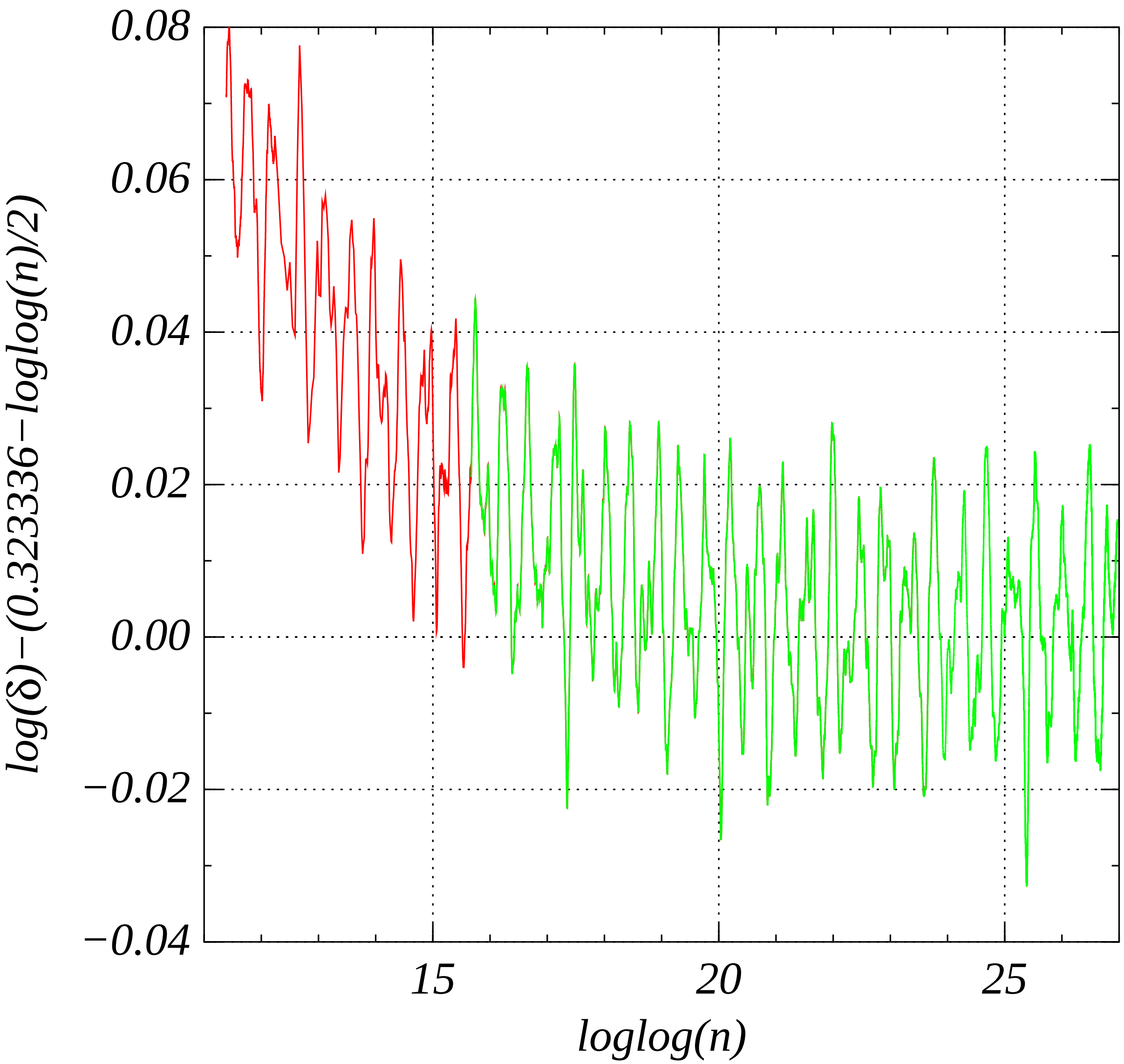
This shows that $\log\delta$ appears to be asymptotically a linear function of $x = \log\log(n)$

Dependence of δ on n



This is the last 100000 values of the previous plot

Deviation of $\log\delta$ from a best-fit line



This shows the difference between $\log\delta$ and a conjectured best-fit line $a - x/2$, where $x \equiv \log\log(n)$

References

[1] J. C. Lagarias *An elementary problem equivalent to the Riemann hypothesis*, Amer. Math. Monthly, **109** (2002), 534-543 www.math.lsa.umich.edu/∼lagarias/doc/elementaryrh.ps

[2] L. Alaoglu & P. Erdős *On highly composite and similar numbers*, Trans. Amer. Math. Soc. **56** (1944) 448-469

[3] G. Robin *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. pures appl. **63** (1984) 187-213

[4] S. Ramanujan *Highly composite numbers. Annotated and with a foreword by J.-L. Nicholas and G. Robin*, Ramanujan J. **1** (1997) 119-153

[5] J.-L. Nicolas *Ordre maximal d'un élément du groupe des permutations et highly composite numbers*, Bull. Math. Soc Fr. **97** (1969), 129-191

[6] P. Erdős and J.-L. Nicolas *Répartition des nombres superabondants*, Bull. Math. Soc Fr. **103** (1975), 65-90