Optimal railway-trip planning

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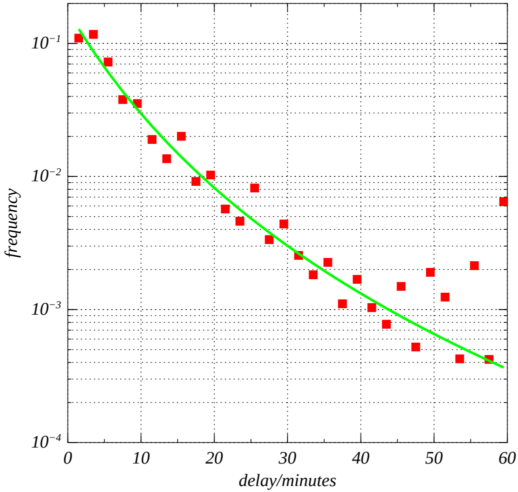
BT Research - Adastral Park

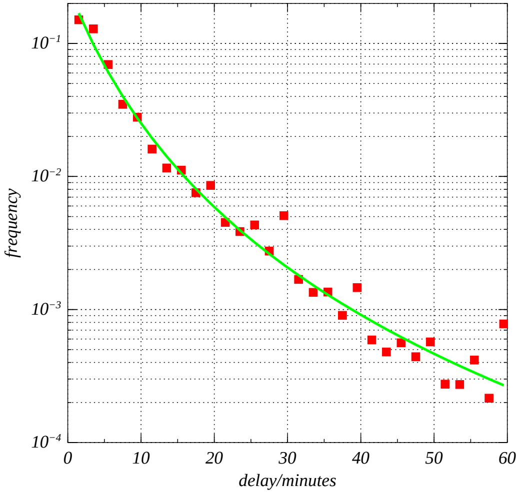
Coventry, all departures

q=1.215
β=0.217



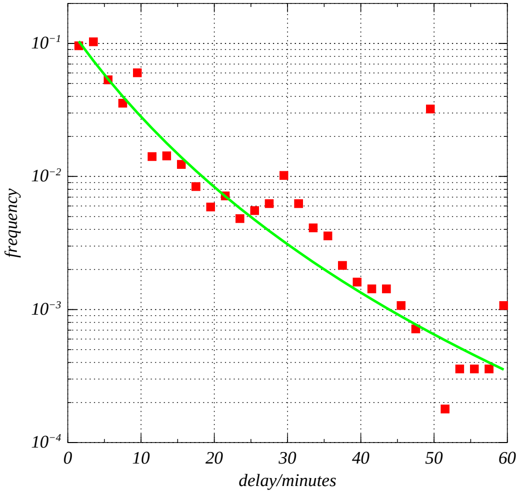
Birmingham, all departures

q=1.263
β=0.329



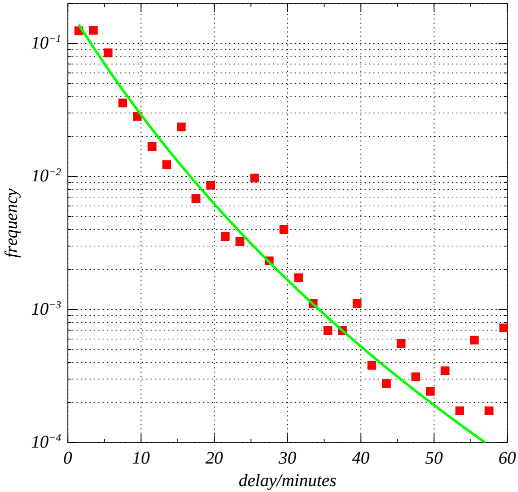
Leicester, all departures

$q=1.186$
$\beta=0.183$



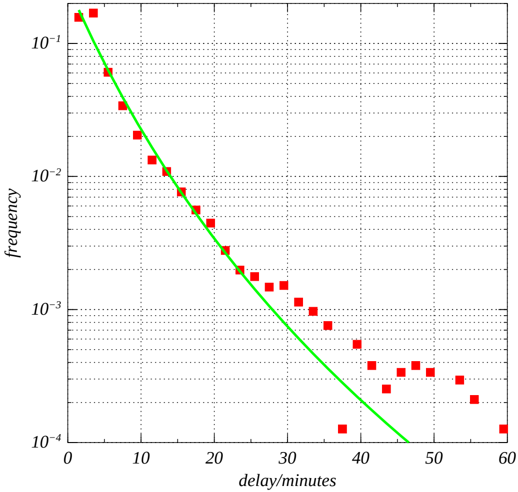
Coventry to Birmingham

q=1.111
β=0.207



Coventry to Euston

$q = 1.127$
$\beta = 0.294$



The $q-$exponential law

- Exponential law: $f_\beta(t) \propto \exp(-\beta t)$
- $e_{q,\beta}(x) := (1/Z)(1+\beta(q-1)x)^{1/(1-q)}$, $\beta > 0, 1 < q < 2$
- $Z := \frac{1}{\beta(2-q)}$
- mean $\mu := \frac{1}{\beta(3-2q)}$
- $\lim_{q \to 1} e_{q,\beta}(t) = \exp(-\beta t)/Z$
- large q gives a power-law (long tail)

The problem loosely stated

- Given: a transport network with timetabled services on each edge (leg)
- Given: a model of the distribution of delays
- Given: a probabilistic optimality criterion (such as *the chance of a final delay more than 10 minutes is less than 5%*)
- To find: some routes satisfying the criterion
- To find: the latest departure time satisfying the criterion

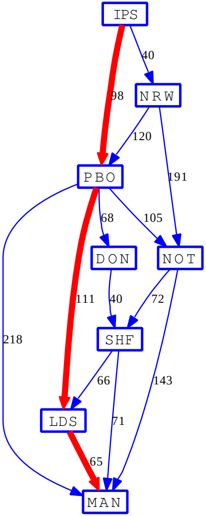
The problem formalized

- Given: a weighted digraph g, a timetable $\tau(n_0, n_1)$ for each arc $(n_0, n_1) \in g$, an arrival time α, and parameters $\tau > 0, \epsilon > 0$.
- To find: a route ρ and maximal departure time t such that Prob[arrival after $\alpha + \tau] < \epsilon$

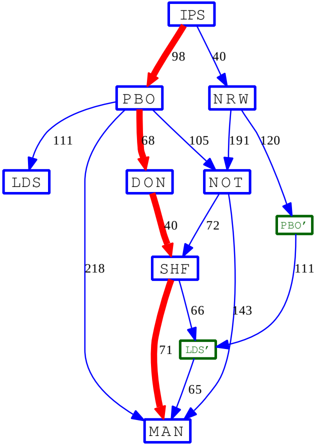
Stochastically short paths

- Given a graph with RVs as edge weights and two nodes, we could:
  - minimize expected time to travel between the nodes
  - find a route which maximizes the probability that it is shortest
  - find the route of shortest mean time, subject to some condition on the variance
  - ...

Short paths in a weighted digraph

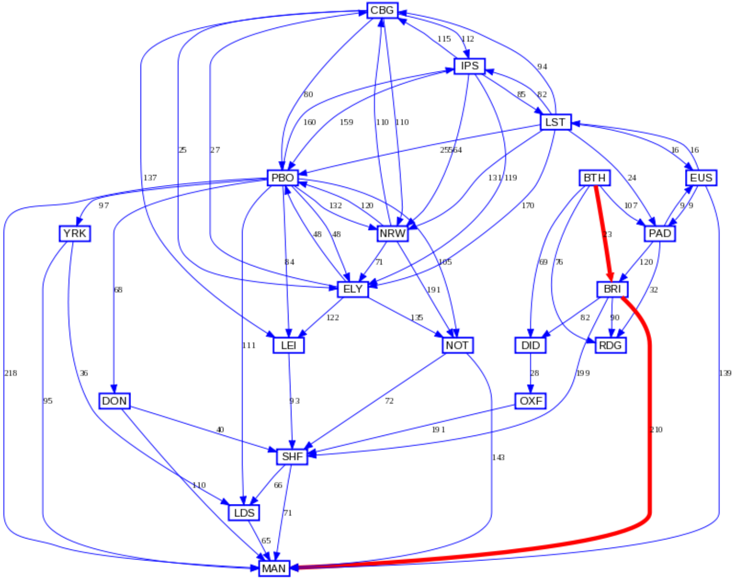


time=275



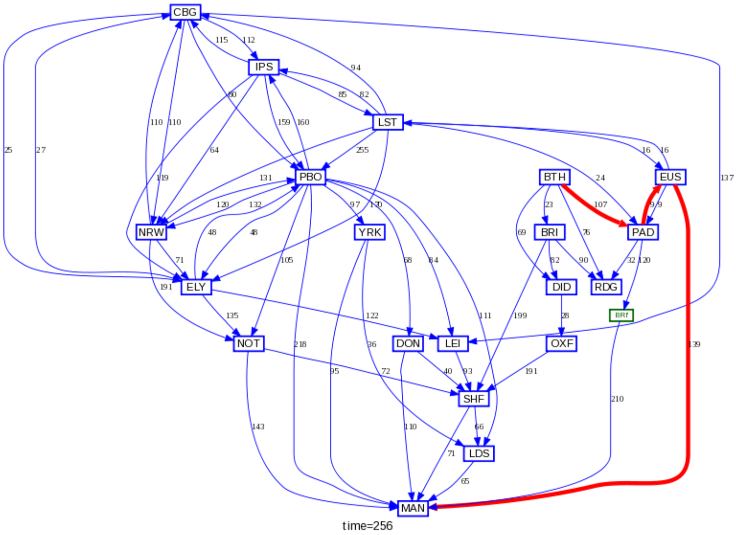
time=278

Bath to Manchester, shortest mean time

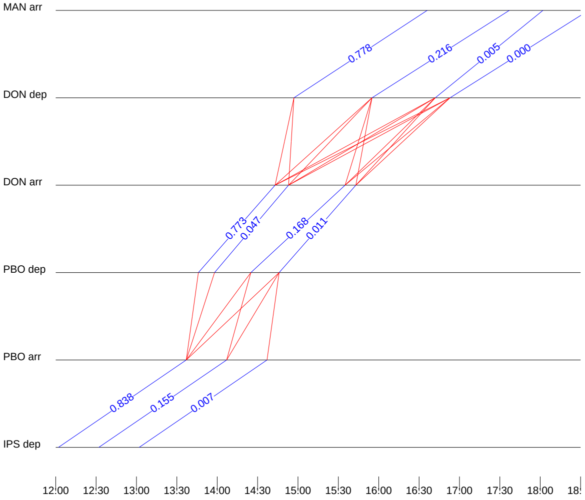


time=233

Bath to Manchester, second shortest mean time



Markov model of train transitions



Markov model of train transitions

- Idea: the states of the system are the particular trains the passenger is on
- Train-changing rule: the passenger always takes the first train going along his pretermined route (no dynamic recomputation of routes)
- 2-tuple indices: (i, j) $(i \geqslant 0, -\infty < j < \infty)$ means that on leg i, the passenger took train j

General setting for one train transition

- Given: a sequence of independent real-valued RVs X_i $(-\infty < i < \infty)$ such that $\mathrm{supp}(X_i) = (t_i, \infty)$ with t_i a strictly increasing (time) sequence.
- For a given time t find (for each i) the probability $p_i(t)$ that X_i is the smallest value such that $X_i > t$.
- That is, if $j \neq i$, either $X_j < t$, or $X_j > X_i$.

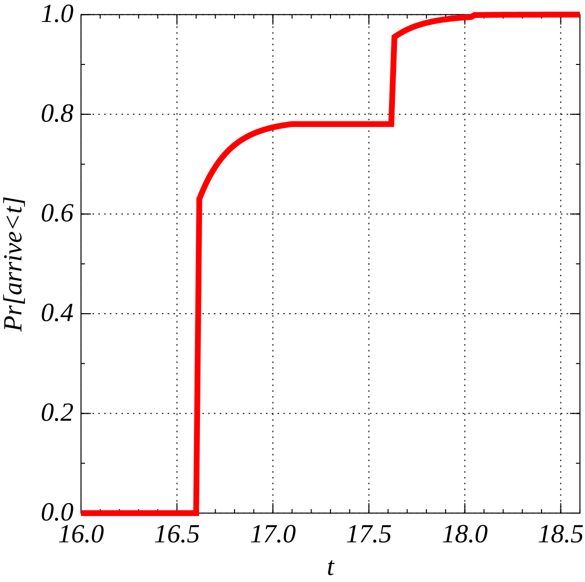
Example for $n=3$

- Let $f_1(x_1)f_2(x_2)f_3(x_3)$ be the joint density.
- Then $p_i(t)=\int\int\int f_1(x_1)f_2(x_2)f_3(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ on a suitable domain.
- e.g. $p_1(t)=\Pr[X_2<t, X_3<t, t<X_1]+\Pr[X_2<t, t<X_1, X_1<X_3]+\Pr[t<X_1, X_1<X_2, X_1<X_3]$
- $\Pr[X_2<t, X_3<t, t<X_1]=\int_{x_2<t}\int_{x_3<t}\int_{t<x_1} f_1(x_1)f_2(x_2)f_3(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$

Special case - shifted exponentials

- $f_i(x_i;t)=H(t>t_i, \lambda_i \exp(-\lambda_i(x-t_i)),0),\ i=1,2,3,\ldots$
- $\Pr[X_1<t<X_0<X_2]=-H(t<d_2,0,d_2<t,-\exp(-\lambda_2(t-d_2))+1)(-H(t<d_1,0,t<d_3,\exp(-\lambda_1(t-d_1))-1,d_3<t,(\lambda_1\exp(-\lambda_1 t+\lambda_1 d_1+\lambda_3 d_3-\lambda_3 t)+\exp(\lambda_1(d_1-d_3))\lambda_3-\lambda_1-\lambda_3)/(\lambda_1+\lambda_3))\exp(\lambda_1 d_3)\lambda_1-H(t<d_1,0,t<d_3,\exp(-\lambda_1(t-d_1))-1,d_3<t,(\lambda_1\exp(-\lambda_1 t+\lambda_1 d_1+\lambda_3 d_3-\lambda_3 t)+\exp(\lambda_1(d_1-d_3))\lambda_3-\lambda_1-\lambda_3)/(\lambda_1+\lambda_3))\exp(\lambda_1 d_3)\lambda_3+\exp(\lambda_1 d_1)\lambda_3-\lambda_1\exp(\lambda_1 d_3)-\lambda_3\exp(\lambda_1 d_3))/(\lambda_1+\lambda_3)\exp(-\lambda_1 d_3)$
- Here $H(C,T,F)$ is T if C is true, else F

IPS→PBO→DON→MAN dep. 1200



IPS→PBO→DON→MAN dep. 12:00

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IPS 12:02 -> PBO 13:37 0.8380
   PBO 13:46 -> DON 14:43 0.7731
      DON 14:57 -> MAN 16:36 p=0.7377
      DON 15:55 -> MAN 17:37 p=0.0353
   PBO 13:58 -> DON 14:53 0.0466
      DON 14:57 -> MAN 16:36 p=0.0405
      DON 15:55 -> MAN 17:37 p=0.0061
   PBO 14:25 -> DON 15:35 0.0173
      DON 15:55 -> MAN 17:37 p=0.0169
      DON 16:42 -> MAN 18:02 p=0.0004
   PBO 14:46 -> DON 15:43 0.0009
      DON 15:55 -> MAN 17:37 p=0.0009
IPS 12:32 -> PBO 14:07 0.1551
   PBO 14:25 -> DON 15:35 0.1505
      DON 15:55 -> MAN 17:37 p=0.1468
      DON 16:42 -> MAN 18:02 p=0.0036
   PBO 14:46 -> DON 15:43 0.0041
      DON 15:55 -> MAN 17:37 p=0.0039
      DON 16:42 -> MAN 18:02 p=0.0002
```

Algorithm

- Phase 0: find set P of the 3 or 4 paths of shortest mean time
- Phase 1: for each path $p \in P$, and for a given start time, propagate all probabilities through the graph using the Markov model
- Compute probability $\rho = \text{Prob}[\text{arrival after } \alpha + \tau]$
- If $\rho > \epsilon$, repeat with an earlier start time.

IPS→MAN arr. 19:00

```
Iter 1:  Probability of arriving by 19:00 is 99.9%
Ipswich            12:02 -> Peterborough          13:37
Peterborough       14:56 -> Doncaster             15:55
Doncaster          16:42 -> Manchester Piccadilly  18:02

Iter 2:  Probability of arriving by 19:00 is 98.3%
Ipswich            12:02 -> Peterborough          13:37
Peterborough       14:56 -> Doncaster             15:55
Doncaster          16:53 -> Sheffield             17:20
Sheffield          17:40 -> Manchester Piccadilly  18:36

Iter 3:  Probability of arriving by 19:00 is 95.7%
Ipswich            12:02 -> Peterborough          13:37
Peterborough       14:56 -> Doncaster             15:55
Doncaster          17:01 -> Leeds                 17:36
Leeds              17:55 -> Manchester Piccadilly  18:49
```

Reference

K M Briggs & C Beck, *Modelling train delays with q-exponential functions* Physica **A 378**, 498–504 (2007).